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Holomorphic triangles and invariants for smooth four-manifolds

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Abstract

In earlier articles, the authors introduced invariants for closed, oriented three-manifolds, defined using a variant of Lagrangian Floer homology in the symmetric products of Riemann surfaces. The aim of this article is to introduce invariants of oriented, smooth four-manifolds, built using these Floer homology groups. This four-dimensional theory also endows the corresponding three-dimensional theories with additional structure: an absolute grading of certain of its Floer homology groups.

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1. Introduction

The aim of this article is to introduce invariants of oriented, smooth four-manifolds, built using the “Heegaard Floer homology” theories defined in [16,15]. The key ingredient in this construction is an invariant associated to cobordisms between three-manifolds,

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which is built using the holomorphic triangle construction. A related construction endows certain Heegaard Floer homology groups with an absolute grading taking values in the rational numbers. Applications of this absolute grading can be found in [14]. Moreover, a suitable adaptation of the cobordism invariant gives rise to a smooth four-manifold invariant defined for four-manifolds with $b_2^+(X) > 1$. We turn now to a more detailed summary.

1.1. Invariants of cobordisms

In [16,15], we defined Floer homology theories for oriented three-manifolds equipped with Spin^c structures, $HF^-(Y, t)$, $HF^\infty(Y, t)$, and $HF^+(Y, t)$. (There is a fourth invariant, $\widehat{HF}(Y, t)$, which we will not discuss in this introduction.) These invariants fit into a long exact sequence

$$\dots \longrightarrow HF^-(Y, t) \xrightarrow{i} HF^\infty(Y, t) \xrightarrow{\pi} HF^+(Y, t) \xrightarrow{\delta} \dots$$

We abbreviate this long exact sequence $HF^\circ(Y, t)$. (A rapid overview of the general properties of these constructions, together with a proof of their topological invariance in a stronger form, is given in Section 2.) Recall that there is another associated three-manifold invariant, denoted

$$HF_{\text{red}}^+(Y, s) = \text{Coker}(\pi) \cong \text{Ker}(i) = HF_{\text{red}}^-(Y, s).$$

The isomorphism between $HF_{\text{red}}^+(Y, s) \cong HF_{\text{red}}^-(Y, s)$ is induced by the coboundary map.

In the first part of the paper, we construct the invariant of a connected cobordism W between two connected three-manifolds Y_1 and Y_2 , defined using the holomorphic triangle construction and a handle-decomposition of W . Specifically, these constructions give rise to a chain map between the chain complexes from Y_1 to Y_2 , whose induced maps on homology are invariants of W (i.e. they are independent of the handle decomposition).

Theorem 1.1. *The maps on homology induced by a smooth, oriented cobordism W equipped with a Spin^c structure $s \in \text{Spin}^c(W)$ are invariants of the cobordism, inducing a map of long exact sequences:*

$$\begin{array}{ccccccc} \dots & \longrightarrow & HF^-(Y_1, t_1) & \xrightarrow{i_1} & HF^\infty(Y_1, t_1) & \xrightarrow{\pi_1} & HF^+(Y_1, t_1) \longrightarrow \dots \\ & & F_{W,s}^- \downarrow & & F_{W,s}^\infty \downarrow & & F_{W,s}^+ \downarrow \\ \dots & \longrightarrow & HF^-(Y_2, t_2) & \xrightarrow{i_2} & HF^\infty(Y_2, t_2) & \xrightarrow{\pi_2} & HF^+(Y_2, t_2) \longrightarrow \dots, \end{array}$$

(where $t_i \in \text{Spin}^c(Y_i)$ denotes the restriction of s to Y_i), where the vertical maps are uniquely determined up to an overall sign, and all squares are commutative.

The map on long exact sequences induced by the cobordism W and Spin^c structure \mathfrak{s} is abbreviated $F_{W,\mathfrak{s}}^\circ$. Various refinements of the above map, including one using the action of first the homology of W , and one using twisted coefficients, can be found in Section 3.

The maps satisfy certain general properties: duality, conjugation invariance, a blow-up formula, and composition properties. The duality property relates the map induced by the cobordism W , thought of as a cobordism from Y_1 to Y_2 , with the induced map obtained by thinking of W as a cobordism from $-Y_2$ to $-Y_1$. Conjugation invariance sets up an identification between the map induced by W and the Spin^c structure \mathfrak{s} with the same cobordism, equipped with its conjugate Spin^c structure $\bar{\mathfrak{s}}$. The blow-up formula relates the map induced by W , with that induced by the (internal) connected sum of W with $\overline{\mathbb{CP}}^2$ (the complex projective plane, given the opposite of its complex orientation).

The composition law states that if W_1 is a cobordism from Y_1 to Y_2 and W_2 is a cobordism from Y_2 to Y_3 , and we equip W_1 and W_2 with Spin^c structures \mathfrak{s}_1 and \mathfrak{s}_2 respectively (whose restrictions agree over Y_2), then we have the following relationship between the composition of $F_{W_1,\mathfrak{s}_1}^\circ$ with $F_{W_2,\mathfrak{s}_2}^\circ$, and the maps induced by the composite cobordism $W = W_1 \#_{Y_2} W_2$:

$$F_{W_2,\mathfrak{s}_2}^\circ \circ F_{W_1,\mathfrak{s}_1}^\circ = \sum_{\{\mathfrak{s} \in \text{Spin}^c(W) \mid \mathfrak{s}|_{W_1} = \mathfrak{s}_1, \mathfrak{s}|_{W_2} = \mathfrak{s}_2\}} \pm F_{W,\mathfrak{s}}^\circ.$$

All of these properties (and various refinements) are stated precisely in Section 3. The invariants are constructed in Section 4, and the properties are verified in Sections 4–6.

1.2. Absolute gradings

The principles used in construction of the cobordism invariant also allow us to define an absolute \mathbb{Q} -lift of the relative \mathbb{Z} -gradings of the Floer homology groups $HF^\circ(Y, \mathfrak{t})$ for a three-manifold Y equipped with a torsion Spin^c structure \mathfrak{t} . This lift is constructed in Section 7.

The relationship between the cobordism invariant and this absolute grading is codified in the following formula, which holds for any cobordism W from Y_1 to Y_2 , equipped with a Spin^c structure \mathfrak{s} whose restriction to the two boundary components \mathfrak{t}_1 and \mathfrak{t}_2 are both torsion:

$$\widetilde{\text{gr}}(F_{W,\mathfrak{s}}^+(\xi)) - \widetilde{\text{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

(where $\xi \in HF^+(\xi, \mathfrak{t}_1)$ is any homogeneous cohomology class, and $\widetilde{\text{gr}}$ denotes the absolute \mathbb{Q} degree).

1.3. Closed four-manifold invariants

There is a variant of the cobordism invariant which can be defined for cobordisms W with $b_2^+(W) > 1$.

Observe first that if W is a cobordism with $b_2^+(W) > 0$, then the map $F_{W,\mathfrak{s}}^\infty$ induced on HF^∞ is trivial (cf. Lemma 8.2). If we have a cobordism with $b_2^+(W) > 1$, then we can cut W along a three-manifold N , to divide it into two cobordisms W_1 and W_2 , both of which have $b_2^+(W_i) > 0$, in such a way that the map induced by restriction

$$\mathrm{Spin}^c(W) \longrightarrow \mathrm{Spin}^c(W_1) \times \mathrm{Spin}^c(W_2)$$

is injective (i.e. $\delta H^1(Y; \mathbb{Z}) \subset H^2(W)$ is trivial). In view of these remarks, if \mathfrak{s} is a Spin^c structure and \mathfrak{s}_1 and \mathfrak{s}_2 denote its restriction to W_1 and W_2 , respectively, then

$$F_{W_1,\mathfrak{s}_1}^-: HF^-(Y_1, \mathfrak{t}_1) \longrightarrow HF^-(N, \mathfrak{t})$$

factors through the inclusion of $HF_{\mathrm{red}}^-(N, \mathfrak{t}) \hookrightarrow HF^-(N, \mathfrak{t})$, while

$$F_{W_2,\mathfrak{s}_2}^+: HF^+(N, \mathfrak{t}) \longrightarrow HF^+(Y_2, \mathfrak{t}_2)$$

factors through the projection $HF^+(N, \mathfrak{t}) \longrightarrow HF_{\mathrm{red}}^+(N, \mathfrak{t})$. Thus, by using the identification of $HF_{\mathrm{red}}^+(N, \mathfrak{t}) \cong HF_{\mathrm{red}}^-(N, \mathfrak{t})$ in the middle, we can define the “mixed invariant” as a map

$$F_{W,\mathfrak{s}}^{\mathrm{mix}}: HF^-(Y_1, \mathfrak{t}_1) \longrightarrow HF^+(Y_2, \mathfrak{t}_2).$$

When X is a closed four-manifold with $b_2^+(X) > 0$, we can puncture it in two points, and view the resulting object as a cobordism from S^3 to S^3 . By elaborating on the mixed invariant construction in Section 9, we obtain a map

$$\Phi_{\mathfrak{s}}: \mathbb{Z}[U] \otimes \Lambda^*(H_1(X; \mathbb{Z})/\mathrm{Tors}) \longrightarrow \mathbb{Z},$$

which is a homogeneous polynomial of degree

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X))}{4},$$

and well-defined up to sign.

It is, of course, interesting to compare this invariant with the Seiberg–Witten invariant [18]. Indeed, we formulate the following:

Conjecture 1.2. *Let X be a closed, oriented, smooth four-manifold with $b_2^+(X) > 1$. Then, the invariant $\Phi_{X,\mathfrak{s}}$ agrees with the Seiberg–Witten invariant for X in the Spin^c structure \mathfrak{s} .*

The invariant Φ can be used to give more directly topological “gauge-theory free” proofs of some facts about smooth four-manifolds which have been previously established by means of Donaldson polynomials and Seiberg–Witten invariants. We return to calculations of Φ in [12], and its non-vanishing properties for symplectic manifolds, after calculating HF^+ for a number of three-manifolds in [14]. These calculations are based on the surgery long exact sequences from [15], combined with the absolute \mathbb{Q} grading defined in the present article. We content ourselves here with some general properties and vanishing results, all of which have natural analogues in Seiberg–Witten theory.

1.4. Basic properties of the closed invariant

The following is an analogue of Donaldson’s connected sum theorem for his polynomial invariants [4]. Unlike its gauge-theoretic counterpart, the result follows rather directly from the definition of the invariants.

Theorem 1.3. *Let X_1 and X_2 be a pair of smooth, oriented four-manifolds with $b_2^+(X_1), b_2^+(X_2) > 0$. Then, the invariants $\Phi_{X,\mathfrak{s}}$ for the connected sum $X = X_1 \# X_2$ vanish identically, for all Spin^c structures.*

The blow-up formula for the cobordism invariant translates directly into a corresponding blow-up formula for $\Phi_{X,\mathfrak{s}}$ (cf. [5]):

Theorem 1.4. *Let X be a closed, smooth, four-manifold with $b_2^+(X) > 1$, and let $\widehat{X} = X \# \overline{\mathbb{CP}}^2$ be its blowup. Then, for each Spin^c structure $\widehat{\mathfrak{s}} \in \text{Spin}^c(\widehat{X})$, with $d(\widehat{X}, \widehat{\mathfrak{s}}) \geq 0$ we have the relation*

$$\Phi_{\widehat{X},\widehat{\mathfrak{s}}}(U^{\frac{\ell(\ell+1)}{2}} \cdot \xi) = \Phi_{X,\mathfrak{s}}(\xi),$$

where \mathfrak{s} is the Spin^c structure over X which agrees over $X - B^4$ with the restriction of $\widehat{\mathfrak{s}}$, $\xi \in \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{Tors})$ is any element of degree $d(X, \mathfrak{s})$, and ℓ is determined by $\langle c_1(\widehat{\mathfrak{s}}), [E] \rangle = \pm(2\ell + 1)$, where $E \subset \widehat{X}$ is the exceptional sphere.

For three-manifolds, Theorem 7.1 of [15] gives bounds on the Thurston norm in terms the first Chern classes of Spin^c structures for which HF^+ is non-trivial. These “adjunction inequalities” have a straightforward generalization in the four-dimensional context (cf. [10,11,13]).

Theorem 1.5. *Let $\Sigma \subset X$ be a homologically non-trivial embedded surface with genus $g \geq 1$ and with non-negative self-intersection number. Then, for each Spin^c structure*

$\mathfrak{s} \in \text{Spin}^c(X)$ for which $\Phi_{X,\mathfrak{s}} \neq 0$, we have that

$$\langle c_1(\mathfrak{s}), [\Sigma] \rangle + [\Sigma] \cdot [\Sigma] \leq 2g - 2.$$

1.5. Organization

In Section 2, we collect some of the basic properties of the three-manifold invariants defined in [16]. Specifically, we establish a version of naturality for these invariants which will be necessary for our future purposes. In Section 3, we outline the basic properties of the invariants for cobordisms. The invariant is constructed in Section 4; its basic properties are verified in this section and also Sections 5 and 6. In Section 7, we define an absolute \mathbb{Q} lift of the relative \mathbb{Z} -graded homology groups associated to a three-manifold equipped with a torsion Spin^c structure (by which we mean a Spin^c structure whose first Chern class is a torsion cohomology class). In Section 8, we define the mixed invariant for cobordisms. In Section 9, this is used to define the invariant Φ for closed four-manifolds X with $b_2^+(X) > 1$. The properties of this invariant stated above are verified in Section 10.

1.6. Further developments

In [14], we give applications of the absolute \mathbb{Q} -grading to intersection forms of four-manifolds bounding a given three-manifold. These results include a proof of Donaldson's diagonalizability theorem [3] from the point of view of Heegaard Floer homology, and also Frøyshov's results for intersection forms of four-manifolds bounding rational homology three-spheres, see [6]. The interplay between absolute gradings and the surgery long exact sequences as developed in [15] allow one to calculate the groups for a wide class of three-manifolds, as begun in [14]. In [12], we prove a non-triviality theorem for symplectic four-manifolds analogous to a result of Taubes for Seiberg–Witten invariants, see [17].

2. Preliminaries on Floer homology groups and holomorphic triangles

The aim of the present section is to collect some of the basic properties of the three-manifold invariants introduced in [16,15], and to establish their topological invariance in a form which will be useful to us in the subsequent sections. After stating this result, a naturality result for the isomorphism induced by a diffeomorphism, we turn to a general discussion of the maps between the homology groups, as given using Heegaard triples: recalling the four-dimensional interpretation of Heegaard triples in Section 2.2 (cf. Section 8.1 of [16] for a more detailed discussion), and then in Section 2.3 stating the basic properties of the induced maps. After recalling the calculations of the Floer homology of $\#^n(S^1 \times S^2)$, which is basic to much of the present theory, we return to a proof of the naturality result in Section 2.5.

2.1. Floer homology groups

To fix terminology, a *set of attaching circles* $\alpha = \{\alpha_1, \dots, \alpha_g\}$ in a genus g surface Σ is a g -tuple of pairwise disjoint, simple, closed curves in Σ whose homology classes are linearly independent in $H_1(\Sigma; \mathbb{Z})$. A set of attaching circles gives rise to a g -dimensional torus $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g$ inside the g -fold symmetric product of Σ , $\text{Sym}^g(\Sigma)$. A *pointed Heegaard diagram* is a collection $(\Sigma, \alpha, \beta, z)$ where Σ is an oriented two-manifold, α and β are sets of attaching circles in Σ , and z is a fixed reference point in Σ which is disjoint from all of the attaching circles. A Heegaard diagram (Σ, α, β) specifies an oriented three-manifold Y in a natural way.

In fact, given an oriented three-manifold equipped with a metric and a self-indexing Morse function with unique index zero and three critical points, there is an induced Heegaard diagram for Y for which the mid-level set is the Heegaard surface, points lying on α_i (for any $i = 1, \dots, g$) are the points in the mid-level which flow out of the index one critical points under upward gradient flow, and points lying on the β_j are points in the mid-level which flow into the index two critical points. We say that two Heegaard diagrams are *equivalent* if they are associated to two different such Morse functions (and metrics) on the same three-manifold. If two Heegaard diagrams are equivalent, they can be connected by a sequence of isotopies, handleslides, and stabilizations.³ As described in [16], the basepoint z in the pointed Heegaard diagram induces a map from intersection points of the tori \mathbb{T}_α and \mathbb{T}_β in $\text{Sym}^g(\Sigma)$ to Spin^c structures over Y .

Suppose Y is a three-manifold, equipped with a fixed Spin^c structure \mathfrak{t} . To a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y (which satisfy certain additional admissibility hypotheses when $b_1(Y) > 0$, which we will recall shortly) we can associate four chain complexes. These complexes are $\widehat{CF}(\alpha, \beta, \mathfrak{t})$, which is freely generated by intersection points between \mathbb{T}_α and \mathbb{T}_β representing the Spin^c structure \mathfrak{t} , $CF^\infty(\alpha, \beta, \mathfrak{t})$, which is freely generated by pairs consisting of such intersection and an integer, a subcomplex $CF^-(\alpha, \beta, \mathfrak{t})$ where the integer is required to be negative, and a quotient complex $CF^+(\alpha, \beta, \mathfrak{t})$. The boundary maps are defined by counting pseudo-holomorphic Whitney disks in $\text{Sym}^g(\Sigma)$, and so the definition of the boundary maps use some auxiliary data, including a complex structure over Σ , and a one-parameter variation in the induced almost-complex structure over $\text{Sym}^g(\Sigma)$, which we suppress from the notation for the chain complex (indeed, as we have suppressed the surface Σ and its basepoint z). Each of $CF^\infty(\alpha, \beta, \mathfrak{t})$, $CF^-(\alpha, \beta, \mathfrak{t})$ and $CF^+(\alpha, \beta, \mathfrak{t})$ comes equipped with the action of a chain map U which is induced from the map $U[\mathbf{x}, i] = [\mathbf{x}, i - 1]$ on $CF^\infty(\alpha, \beta, \mathfrak{t})$ which decreases the (relative) grading by two.

Recall that when $b_1(Y) > 0$, we work with Heegaard diagrams satisfying additional “admissibility” hypotheses. To state these hypotheses, note that the two-dimensional

³ For us, stabilization means introducing a pair of canceling α - and β -curves which are supported in a torus connected summand of the Heegaard surface. It is easy to see that more general stabilizations—where the canceling curves are allowed to wander over Σ —differ from such stabilizations by a sequence of handleslides.

homology classes in Y can be thought of as two-chains \mathcal{P} in Σ whose local multiplicity at the reference point z is zero, and whose boundary can be represented as a linear combination of cycles representing elements of the α and the β . Such two-chains are called *periodic domains*, and their associated classes in $H_2(Y; \mathbb{Z})$ are denoted $H(\mathcal{P})$. A Heegaard diagram is *strongly t-admissible* for the Spin^c structure t if for each (non-trivial) periodic domain \mathcal{P} for which $\langle c_1(t), H(\mathcal{P}) \rangle = 2n \geq 0$, some local multiplicity in \mathcal{P} is greater than n . When we consider $CF^-(\alpha, \beta, t)$ and $CF^\infty(\alpha, \beta, t)$, we will always be using strongly t -admissible Heegaard diagrams. When working with $\widehat{CF}(\alpha, \beta, t)$ and $CF^+(\alpha, \beta, t)$, it suffices to work with *weakly t-admissible Heegaard diagrams* which are those for which each non-trivial periodic domain \mathcal{P} with $\langle c_1(t), H(\mathcal{P}) \rangle = 0$ has both positive and negative coefficients. Every Heegaard diagram for Y is isotopic to a strongly t -admissible Heegaard diagram (see [16], Section 5).

It is shown in [16,15] that homology groups of these chain complexes— $\widehat{HF}(Y, t)$, $HF^\infty(Y, t)$, $HF^-(Y, t)$, and $HF^+(Y, t)$ —are topological invariants; where the latter three graded groups are thought of as $\mathbb{Z}[U]$ modules. (In particular, they are independent of the complex structure over Σ , the path of almost-complex structures over $\text{Sym}^g(\Sigma)$, and the Heegaard diagrams used in their definition.)

When $b_1(Y) > 0$, these groups have some extra structure. First of all, the groups themselves depend on an additional choice of a coherent orientation system, but there is a canonical such orientation system fixed in Theorem 10.12 of [15], so that is the one we will use (unless otherwise specified). Second, there is an action by the exterior algebra $\Lambda^* H_1(Y; \mathbb{Z})/\text{Tors}$, graded so that $H_1(Y; \mathbb{Z})/\text{Tors}$ decreases degree by one. Third, there are variants of these homology groups, but with coefficients twisted by an arbitrary $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module M , denoted $\widehat{HF}(Y, t, M)$, $\underline{HF}^-(Y, t, M)$, $\underline{HF}^\infty(Y, t, M)$, and $\underline{HF}^+(Y, t, M)$. These are all modules over the group-ring $\mathbb{Z}[H^1(Y; \mathbb{Z})]$. (When M is not the trivial module \mathbb{Z} over $\mathbb{Z}[H^1(Y; \mathbb{Z})]$, the corresponding Floer modules no longer support an action by $\Lambda^* H_1(Y; \mathbb{Z})/\text{Tors}$.)

There are two natural long exact sequences whose existence is guaranteed immediately from the definitions of these groups

$$\dots \longrightarrow \widehat{HF}(Y, t) \xrightarrow{\widehat{\tau}} HF^+(Y, t) \xrightarrow{U} HF^-(Y, t) \longrightarrow \dots \quad (1)$$

and

$$\dots \longrightarrow HF^-(Y, t) \xrightarrow{l} HF^\infty(Y, t) \xrightarrow{\pi} HF^+(Y, t) \longrightarrow \dots \quad (2)$$

(with analogues in the twisted case, as well).

Sometimes, we abbreviate these long exact sequences simply by $HF^\circ(Y, t)$. Both long exact sequences are also topological invariants of Y . We make this topological invariance statement more precise, by organizing the results from [16,15] to prove the following:

Theorem 2.1. *If $(\Sigma, \alpha, \beta, z)$ and $(\Sigma', \alpha', \beta', z')$ are equivalent Heegaard diagrams which are strongly admissible for the Spin^c structure \mathfrak{t} , then there are induced isomorphisms of corresponding long exact sequences:*

$$\begin{array}{ccccccc} \dots & \longrightarrow & HF^-(\alpha, \beta, \mathfrak{t}) & \xrightarrow{\iota} & HF^\infty(\alpha, \beta, \mathfrak{t}) & \xrightarrow{\pi} & HF^+(\alpha, \beta, \mathfrak{t}) \longrightarrow \dots \\ & & \Psi_{\mathfrak{t}}^- \downarrow & & \Psi_{\mathfrak{t}}^\infty \downarrow & & \Psi_{\mathfrak{t}}^+ \downarrow \\ \dots & \longrightarrow & HF^-(\alpha', \beta', \mathfrak{t}) & \xrightarrow{\iota'} & HF^\infty(\alpha', \beta', \mathfrak{t}) & \xrightarrow{\pi'} & HF^+(\alpha', \beta', \mathfrak{t}) \longrightarrow \dots \end{array}$$

(i.e. where each square commutes), where the vertical maps commute with the actions of $\mathbb{Z}[U] \otimes \Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors})$; and also

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widehat{HF}(\alpha, \beta, \mathfrak{t}) & \xrightarrow{\widehat{\iota}} & HF^+(\alpha, \beta, \mathfrak{t}) & \xrightarrow{U} & HF^+(\alpha, \beta, \mathfrak{t}) \longrightarrow \dots \\ & & \widehat{\Psi}_{\mathfrak{t}} \downarrow & & \Psi_{\mathfrak{t}}^+ \downarrow & & \Psi_{\mathfrak{t}}^+ \downarrow \\ \dots & \longrightarrow & \widehat{HF}(\alpha', \beta', \mathfrak{t}) & \xrightarrow{\widehat{\iota}'} & HF^+(\alpha', \beta', \mathfrak{t}) & \xrightarrow{U'} & HF^+(\alpha', \beta', \mathfrak{t}) \longrightarrow \dots, \end{array}$$

which commutes with the action of $\Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors})$. Moreover, the maps $\widehat{\Psi}$, Ψ^- , Ψ^∞ , and Ψ^+ are uniquely determined up to an overall factor of ± 1 .

We return to a proof of Theorem 2.1 in Section 2.5. (See also a version for twisted coefficients in Section 2.6.)

A *pointed Heegaard triple* $(\Sigma, \alpha, \beta, \gamma, z)$ is defined using three sets of attaching circles α , β , and γ , with a reference point $z \in \Sigma$ disjoint from all these attaching circles. By counting holomorphic triangles in $\text{Sym}^g(\Sigma)$ with boundary conditions specified by the three subspaces \mathbb{T}_α , \mathbb{T}_β , and \mathbb{T}_γ , we get maps between tensor products of Floer homology groups. On the other hand, the Heegaard triple and the homotopy classes of triangles both can be interpreted in terms of four-dimensional data. We briefly digress to recall this four-dimensional interpretation (referring the reader to Section 8 of [16] for more details).

2.2. Homotopy classes of triangles, and Spin^c structures on four-manifolds

Given a pointed Heegaard triple we have three g -dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g$, $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g$ and $\mathbb{T}_\gamma = \gamma_1 \times \dots \times \gamma_g$ which are embedded in the g -fold symmetric product of Σ , $\text{Sym}^g(\Sigma)$.

Clearly, the Heegaard triple specifies three three-manifolds $Y_{\alpha, \beta}$, $Y_{\beta, \gamma}$, and $Y_{\alpha, \gamma}$ with Heegaard diagrams given by using the corresponding pairs of g -tuples of attaching circles. Indeed, the Heegaard triple specifies a four-manifold $X_{\alpha, \beta, \gamma}$ which bounds these three three-manifolds. We recall the construction presently.

Let Δ denote the two-simplex, with vertices $v_\alpha, v_\beta, v_\gamma$ labeled clockwise, and let e_i denote the edge v_j to v_k , where $\{i, j, k\} = \{\alpha, \beta, \gamma\}$. Then, we form the identification space

$$X_{\alpha, \beta, \gamma} = \frac{(\Delta \times \Sigma) \coprod (e_\alpha \times U_\alpha) \coprod (e_\beta \times U_\beta) \coprod (e_\gamma \times U_\gamma)}{(e_\alpha \times \Sigma) \sim (e_\alpha \times \partial U_\alpha), (e_\beta \times \Sigma) \sim (e_\beta \times \partial U_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial U_\gamma)}.$$

Over the vertices of Δ , this space has corners, which can be naturally smoothed out to obtain a smooth, oriented, four-dimensional cobordism between the three-manifolds $Y_{\alpha, \beta}$, $Y_{\beta, \gamma}$, and $Y_{\alpha, \gamma}$ as claimed. Indeed, under the natural orientation conventions implicit in the above description,

$$\partial X_{\alpha, \beta, \gamma} = -Y_{\alpha, \beta} - Y_{\beta, \gamma} + Y_{\alpha, \gamma}.$$

Recall that the two-dimensional homology of X corresponds to the set of *triply periodic domains* for the Heegaard triples—these are two-chains in Σ (which can be represented by maps $\Phi: F \rightarrow \Sigma$ where F is a two-dimensional manifold-with-boundary) with multiplicity zero at the reference point z , and whose boundaries are linear combinations of curves appearing in the tuples α , β , and γ . Such chains in turn correspond uniquely to the integer relations they give amongst the homology classes coming from the α , β , and γ . Given such a two-chain in Σ , the associated homology class $H(\mathcal{P})$ in $H_2(X; \mathbb{Z})$ can be concretely thought of as follows. We can extend $\Phi: F \rightarrow \Sigma$, to a map

$$\widehat{\Phi}: \widehat{F} \rightarrow X,$$

by viewing Φ as a map into some copy of Σ times an interior point $x \in \Delta$, and then attaching cylinders connecting ∂F to a collection of α , β , and γ -circles occurring in copies of Σ on the α , β and γ -edges of $\Delta \times \Sigma$. Then, we cap off these boundary components in the corresponding U_α , U_β and U_γ handlebodies, to get a map from a closed surface into X . This closed surface represents the homology class $H(\mathcal{P})$.

Fix intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. A map

$$u: \Delta \rightarrow \text{Sym}^g(\Sigma)$$

with the boundary conditions that $u(v_\gamma) = \mathbf{x}$, $u(v_\alpha) = \mathbf{y}$, and $u(v_\beta) = \mathbf{w}$, and $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$ is called a *Whitney triangle connecting \mathbf{x} , \mathbf{y} , and \mathbf{w}* . There is a naturally defined map

$$\varepsilon: (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{T}_\beta \cap \mathbb{T}_\gamma) \times (\mathbb{T}_\alpha \cap \mathbb{T}_\gamma) \rightarrow H_1(X; \mathbb{Z})$$

with the property that $\varepsilon(\mathbf{x}, \mathbf{y}, \mathbf{w}) = 0$ if and only if there is a Whitney triangle connecting \mathbf{x} , \mathbf{y} , and \mathbf{w} . Two Whitney triangles are said to be homotopic if the maps are homotopic

through maps which are all Whitney triangles. For fixed \mathbf{x} , \mathbf{y} , and \mathbf{w} , let $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ denote the space of homotopy classes of Whitney triangles connecting \mathbf{x} , \mathbf{y} , and \mathbf{w} . Any two elements of this space $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ have a naturally associated difference, which is a triply periodic domain (after we subtract off a sufficient multiple of Σ). When $g > 2$, this sets up an affine isomorphism, whenever $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ is non-empty,

$$\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \cong \mathbb{Z} \oplus H_2(X_{\alpha, \beta, \gamma}; \mathbb{Z}),$$

where the first factor is given by the local multiplicity at the reference point z . (When $g \leq 2$, and $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ is non-empty, then there is still a map from isomorphism $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ to $H_2(X_{\alpha, \beta, \gamma}; \mathbb{Z})$.)

As explained in Section 8 of [16], a Whitney triangle can be used to construct a singular two-plane field in X whose underlying Spin^c structure is independent of the choices made in its construction, giving rise to a map

$$\mathfrak{s}_z: \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \longrightarrow \text{Spin}^c(X),$$

where $\mathfrak{s}_z(\psi)$ restricts to $\mathfrak{s}_z(\mathbf{x})$, $\mathfrak{s}_z(\mathbf{y})$, and $\mathfrak{s}_z(\mathbf{w})$ at the three boundary components.

There is a splicing map

$$\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \times \pi_2(\mathbf{x}', \mathbf{x}) \times \pi_2(\mathbf{y}', \mathbf{y}) \times \pi_2(\mathbf{w}, \mathbf{w}') \longrightarrow \pi_2(\mathbf{x}', \mathbf{y}', \mathbf{w}'),$$

which we denote simply by addition. We have the following result from [15]:

Lemma 2.2. *Fix $\mathbf{x}, \mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y}, \mathbf{y}' \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{w}, \mathbf{w}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. Two homotopy classes $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ and $\psi' \in \pi_2(\mathbf{x}', \mathbf{y}', \mathbf{w}')$ induce the same Spin^c structure if and only if there are homotopy classes $\phi_1 \in \pi_2(\mathbf{x}', \mathbf{x})$, $\phi_2 \in \pi_2(\mathbf{y}', \mathbf{y})$, and $\phi_3 \in \pi_2(\mathbf{w}, \mathbf{w}')$ with*

$$\psi' = \psi + \phi_1 + \phi_2 + \phi_3.$$

2.3. Holomorphic triangles

We recall the maps induced by the holomorphic triangle construction. Except, of course, for the interpretation of Whitney triangles using four-manifolds, the results we recall here are modeled on corresponding constructions first proposed by Donaldson in Lagrangian Floer homology (see [2,7]).

We begin with a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, z)$, and fix a Spin^c structure \mathfrak{s} over $X_{\alpha, \beta, \gamma}$. Under suitable admissibility hypotheses, there are chain maps:

$$f^\circ(\cdot, \mathfrak{s}): CF^\circ(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}_{\alpha, \beta}) \otimes CF^{\leq 0}(\boldsymbol{\beta}, \gamma, \mathfrak{s}_{\beta, \gamma}) \longrightarrow CF^\circ(\boldsymbol{\alpha}, \gamma, \mathfrak{s}_{\alpha, \gamma}),$$

where $\mathfrak{s}_{\xi,\eta}$ is the restriction of \mathfrak{s} to $Y_{\xi,\eta}$ and where $CF^{\leq 0} \cong CF^-$ is the chain complex generated by pairs $[\mathbf{x}, j]$ with $j \leq 0$, given by the formula:

$$f^\circ([\mathbf{x}, i] \otimes [\mathbf{y}, j]; \mathfrak{s}) = \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \mid \mathfrak{s}_z(\psi) = \mathfrak{s}, \mu(\psi) = 0\}} \left(\# \mathcal{M}(\psi) \right) \cdot [\mathbf{w}, i + j - n_z(\psi)],$$

where $\mathcal{M}(\psi)$ denotes the moduli space of pseudo-holomorphic triangles in the homotopy class ψ , and $\mu(\psi)$ denotes its expected dimension. There is also a variant

$$f^\leq(\cdot, \mathfrak{s}): CF^{\leq 0}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}_{\alpha,\beta}) \otimes CF^{\leq 0}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \mathfrak{s}_{\beta,\gamma}) \longrightarrow CF^{\leq 0}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathfrak{s}_{\alpha,\gamma})$$

defined analogously.

The admissibility hypothesis relevant for the above maps is the \mathfrak{s} -strong admissibility for the Heegaard triple ([16], Definition 8.8). Specifically, we require that for each non-trivial triply periodic domain which can be written as a sum of doubly periodic domains

$$\mathcal{P} = \mathcal{D}_{\alpha,\beta} + \mathcal{D}_{\beta,\gamma} + \mathcal{D}_{\alpha,\gamma}$$

with the property that

$$\langle c_1(\mathfrak{s}_{\alpha,\beta}), H(\mathcal{D}_{\alpha,\beta}) \rangle + \langle c_1(\mathfrak{s}_{\beta,\gamma}), H(\mathcal{D}_{\beta,\gamma}) \rangle + \langle c_1(\mathfrak{s}_{\alpha,\gamma}), H(\mathcal{D}_{\alpha,\gamma}) \rangle = 2n \geq 0,$$

we have some local multiplicity of \mathcal{P} which is greater than n .

Before stating the relevant invariance properties of these constructions, we recall some of the notation involved for isotopies. In Section 5 of [15], we achieved admissibility by using *special isotopies*—isotopies which never cross the basepoint z , and which are realized as exact Hamiltonian isotopies in Σ . Now, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, z)$ are strongly \mathfrak{s} -admissible Heegaard diagrams, and there is a special isotopy connecting $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}'$, then the special isotopy induces an isomorphism

$$\Gamma_{\boldsymbol{\alpha}', \boldsymbol{\alpha}; \boldsymbol{\beta}}: HF^\circ(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \longrightarrow HF^\circ(\boldsymbol{\alpha}', \boldsymbol{\beta}, \mathfrak{s})$$

defined by counting holomorphic disks with time-dependent boundary conditions; similarly, a special isotopy from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$ induces an isomorphism

$$\Gamma_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}'}: HF^\circ(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \longrightarrow HF^\circ(\boldsymbol{\alpha}, \boldsymbol{\beta}', \mathfrak{s})$$

Theorem 2.3. *Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is an admissible Heegaard triple for the Spin^c structure \mathfrak{s} . Then, the induced maps on homology*

$$F^\circ(\cdot, \mathfrak{s}): HF^\circ(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}_{\alpha,\beta}) \otimes HF^{\leq 0}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \mathfrak{s}_{\beta,\gamma}) \longrightarrow HF^\circ(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathfrak{s}_{\alpha,\gamma}),$$

satisfy the following properties:

- $F^\circ(\cdot, \mathfrak{s})$ is independent of the choices of complex structures and perturbations used in the definition of the moduli spaces of triangles.
- $F^\circ(\cdot, \mathfrak{s})$ commutes with the action by $\mathbb{Z}[U]$.
- $F^\circ(\cdot, \mathfrak{s})$ is invariant under special isotopies, in the sense that if

$$\Gamma_{\alpha, \beta, \beta'}: HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \longrightarrow HF^\circ(\alpha, \beta', \mathfrak{s}_{\alpha, \beta'})$$

and

$$\Gamma_{\alpha', \alpha; \beta}: HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \longrightarrow HF^\circ(\alpha', \beta, \mathfrak{s}_{\alpha', \beta})$$

(where $\mathfrak{s}_{\alpha, \beta'} = \mathfrak{s}_{\alpha, \beta} = \mathfrak{s}_{\alpha', \beta}$ denotes the restrictions of \mathfrak{s} to the boundary component) are the isomorphisms induced by some isotopies from the α to the α' , and β to β' , then the following diagrams commute:

$$\begin{array}{ccc} HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_{\beta, \gamma}) & \xrightarrow{F^\circ(\cdot, \mathfrak{s})} & HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma}) \\ \Gamma_{\alpha', \alpha; \beta} \otimes \text{Id} \downarrow & & \downarrow \Gamma_{\alpha', \alpha; \gamma} \\ HF^\circ(\alpha', \beta, \mathfrak{s}_{\alpha, \beta}) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_{\beta, \gamma}) & \xrightarrow{F^\circ(\cdot, \mathfrak{s})} & HF^\circ(\alpha', \gamma, \mathfrak{s}_{\alpha, \gamma}) \end{array}$$

and

$$\begin{array}{ccc} HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_{\beta, \gamma}) & \xrightarrow{F^\circ(\cdot, \mathfrak{s})} & HF^\circ(\alpha, \gamma) \\ \Gamma_{\alpha, \beta, \beta'} \otimes \Gamma_{\beta', \beta, \gamma} \downarrow & & \downarrow \text{Id} \\ HF^\circ(\alpha, \beta', \mathfrak{s}_{\alpha, \beta'}) \otimes HF^{\leq 0}(\beta', \gamma, \mathfrak{s}_{\beta', \gamma}) & \xrightarrow{F^\circ(\cdot, \mathfrak{s})} & HF^\circ(\alpha, \gamma). \end{array}$$

The proof can be found in [15]: the existence is shown in Theorem 8.12 of [16]; independence of complex structures and isotopy invariance are, respectively, Propositions 8.13 and 8.14 from that paper.

The holomorphic triangle construction also satisfies an associativity property which we state separately. For associativity, we use pointed Heegaard quadruples

$$(\Sigma, \alpha, \beta, \gamma, \delta, z), \quad (3)$$

to which we can associate a four-manifold $X_{\alpha, \beta, \gamma, \delta}$ defined using a square rather than a triangle. We assume that the four-manifold $X_{\alpha, \beta, \gamma, \delta}$ satisfies the additional hypothesis that

$$\delta H^1(Y_{\beta, \delta})|Y_{\alpha, \gamma} = 0 \quad \text{and} \quad \delta H^1(Y_{\alpha, \gamma})|Y_{\beta, \delta} = 0. \quad (4)$$

In this case, one can formulate a strong admissibility hypothesis, which depends on a $\delta H^1(Y_{\beta,\delta}) + \delta H^1(Y_{\alpha,\gamma})$ -orbit \mathfrak{S} of a Spin^c structure in $\text{Spin}^c(X_{\alpha,\beta,\gamma,\delta})$ (see Section 8.4 of [16] for the definition). The topological hypotheses guarantee that strong \mathfrak{s} -admissibility can always be achieved for a Heegaard quadruple.

Remark 2.4. The topological hypothesis of Eq. (3) is automatically satisfied if in our Heegaard quadruple, there are two consecutive g -tuples of circles which span the same subspace of $H_1(\Sigma; \mathbb{Z})$. For instance, if the subspaces spanned by γ and δ coincide, then we can express each (β, δ) -periodic domain as a sum of (γ, δ) and (β, γ) -periodic domains (which ensures that $\delta H^1(Y_{\beta,\delta}) = 0$; and similarly, (α, γ) -periodic domains can be expressed as a sum of (γ, δ) - and (δ, α) -periodic domains. In fact, this guarantees that $\delta H^1(Y_{\beta,\delta}) + \delta H^1(Y_{\alpha,\gamma}) = 0$.

Theorem 2.5. *Let $(\Sigma, \alpha, \beta, \gamma, \delta, z)$ be a pointed Heegaard quadruple which is strongly \mathfrak{S} -admissible, where \mathfrak{S} is a $\delta H^1(Y_{\beta,\delta}) + \delta H^1(Y_{\alpha,\gamma})$ -orbit in $\text{Spin}^c(X_{\alpha,\beta,\gamma,\delta})$. Then, we have*

$$\begin{aligned} & \sum_{\mathfrak{s} \in \mathfrak{S}} F_{\alpha,\gamma,\delta}^{\circ}(F_{\alpha,\beta,\gamma}^{\circ}(\xi_{\alpha,\beta} \otimes \theta_{\beta,\gamma}; \mathfrak{s}_{\alpha,\beta,\gamma}) \otimes \theta_{\gamma,\delta}; \mathfrak{s}_{\alpha,\gamma,\delta}) \\ &= \sum_{\mathfrak{s} \in \mathfrak{S}} F_{\alpha,\beta,\delta}^{\circ}(\xi_{\alpha,\beta} \otimes F_{\beta,\gamma,\delta}^{\leq 0}(\theta_{\beta,\gamma} \otimes \theta_{\gamma,\delta}; \mathfrak{s}_{\beta,\gamma,\delta}); \mathfrak{s}_{\alpha,\beta,\delta}), \end{aligned}$$

where $\xi_{\alpha,\beta} \in HF^{\circ}(\alpha, \beta, \mathfrak{s}_{\alpha,\beta})$, $\theta_{\beta,\gamma}$ and $\theta_{\gamma,\delta}$ lie in $HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_{\beta,\gamma})$ and $HF^{\leq 0}(\gamma, \delta, \mathfrak{s}_{\gamma,\delta})$, respectively.

The above is a special case of Theorem 8.16 of [16].

There is a variant of the chain maps $F_{\alpha,\beta,\gamma}^{\circ}$ which incorporates an action of the first homology of $X_{\alpha,\beta,\gamma}$.

First of all, we claim that the map

$$H_1 \left(Y_{\alpha,\gamma} \coprod Y_{\beta,\gamma} \coprod Y_{\alpha,\gamma}; \mathbb{Z} \right) / \text{Tors} \longrightarrow H_1(X_{\alpha,\beta,\gamma}; \mathbb{Z}) / \text{Tors}$$

is surjective. Given $h \in H_1(X_{\alpha,\beta,\gamma}; \mathbb{Z}) / \text{Tors}$, we let

$$(h_{\alpha,\beta}, h_{\beta,\gamma}, h_{\alpha,\gamma}) \in H_1 \left(Y_{\alpha,\gamma} \coprod Y_{\beta,\gamma} \coprod Y_{\alpha,\gamma}; \mathbb{Z} \right) / \text{Tors}$$

be an element which maps to it. We then define

$$\begin{aligned} & F_{\alpha,\beta,\gamma}^{\circ}(h \otimes \xi_{\alpha,\beta} \otimes \theta_{\beta,\gamma}) \\ &= F_{\alpha,\beta,\gamma}^{\circ}((h_{\alpha,\beta} \cdot \xi_{\alpha,\beta}) \otimes \theta_{\beta,\gamma}) + F_{\alpha,\beta,\gamma}^{\circ}(\xi_{\alpha,\beta} \otimes (h_{\beta,\gamma} \cdot \theta_{\beta,\gamma})) - h_{\alpha,\gamma} \cdot F_{\alpha,\beta,\gamma}^{\circ}(\xi_{\alpha,\beta} \otimes \theta_{\beta,\gamma}), \end{aligned} \tag{5}$$

where the actions on the right-hand side all represent the actions of H_1 of the three-manifolds on their corresponding Floer groups.

Lemma 2.6. *Assume that $HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma})$ has no torsion (over \mathbb{Z}). Then, the action defined in Eq. (5) induces a map*

$$F_{\alpha, \beta, \gamma}^\circ(\cdot, \mathfrak{s}): \Lambda^*(H_1(X_{\alpha, \beta, \gamma}; \mathbb{Z})/\text{Tors}) \otimes HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_{\beta, \gamma}) \longrightarrow HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma}).$$

Proof. Let $X = X_{\alpha, \beta, \gamma}$. We have natural isomorphisms

$$\begin{aligned} H_1(Y_{\alpha, \beta})/\text{Tors} &\cong \text{Hom}(H_2(Y_{\alpha, \beta}), \mathbb{Z}) \\ H_1(X)/\text{Tors} &\cong \text{Hom}(H_3(X, \partial X), \mathbb{Z}), \end{aligned}$$

where all homology groups above (and throughout this proof) are understood to use integral coefficients. Note that $H_3(\partial X) \longrightarrow H_3(X)$ surjects, and $H_2(X)$ has no torsion. Thus, we can dualize a portion of the Mayer–Vietoris sequence for the pair $(X, \partial X)$ to give

$$\text{Hom}(H_2(\partial X), \mathbb{Z}) \xrightarrow{\delta^*} \text{Hom}(H_3(X, \partial X), \mathbb{Z}) \longrightarrow 0,$$

establishing surjectivity of the map on H_1/Tors . Note that the image of $\text{Hom}(H_2(X), \mathbb{Z})$ has finite index in the kernel of δ^* .

We verify that the action by $(h_{\alpha, \beta}, h_{\beta, \gamma}, h_{\alpha, \gamma})$ is trivial on the image of $\text{Hom}(H_2(X), \mathbb{Z})$. To this end, recall that there are identifications

$$\begin{aligned} \text{Hom}(H_2(Y_{\alpha, \beta}), \mathbb{Z}) &\cong H^1(\mathbb{T}_\alpha) \oplus H^1(\mathbb{T}_\beta)/H^1(\text{Sym}^g(\Sigma)), \\ \text{Hom}(H_2(X), \mathbb{Z}) &\cong H^1(\mathbb{T}_\alpha) \oplus H^1(\mathbb{T}_\beta) \oplus H^1(\mathbb{T}_\gamma)/H^1(\text{Sym}^g(\Sigma)). \end{aligned}$$

In particular, those elements of $\text{Hom}(H_2(\partial X), \mathbb{Z})$ which come from $\text{Hom}(H_2(X), \mathbb{Z})$ can be represented by three constraints, one in each of $H^1(\mathbb{T}_\alpha)$, $H^1(\mathbb{T}_\beta)$, and $H^1(\mathbb{T}_\gamma)$. By pushing the constraints out to the ends of the triangles, it follows that the action triples $(h_{\alpha, \beta}, h_{\beta, \gamma}, h_{\alpha, \gamma})$ coming from $\text{Hom}(H_2(X), \mathbb{Z})$ is trivial.

We have just constructed a map

$$\frac{\text{Hom}(H_2(X); \mathbb{Z})}{\text{Hom}(H_2(\partial X); \mathbb{Z})} \otimes HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \longrightarrow HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma}).$$

Since $\text{Ker} \delta^*/\text{Hom}(H_2(\partial X); \mathbb{Z})$ is torsion, and $HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma})$ is torsion-free, it follows that this map extends to

$$\text{Hom}(H_3(X, \partial X), \mathbb{Z}) \otimes HF^\circ(\alpha, \beta, \mathfrak{s}_{\alpha, \beta}) \longrightarrow HF^\circ(\alpha, \gamma, \mathfrak{s}_{\alpha, \gamma}).$$

We have thus constructed the action by $\text{Hom}(H_3(X, \partial X), \mathbb{Z}) \cong \Lambda^1(H_1(X)/\text{Tors})$. This extends to an action of the full exterior algebra: this follows from the corresponding fact for the Floer homologies themselves. Similarly, independence of the action from the auxiliary data (paths of complex structures, isotopies, etc.) follows from the corresponding facts for the three bounding three-manifolds, as well. \square

2.4. Preliminaries on $\#^n(S^1 \times S^2)$.

Let s_0 be the Spin^c structure over $\#^n(S^1 \times S^2)$ whose first Chern class vanishes. As in Section 9 of [16] (cf. also Proposition 6.4 of [15]),

$$\begin{aligned}\widehat{HF}(\#^n(S^1 \times S^2), s_0) &\cong \wedge^* H^1(\#^n(S^1 \times S^2); \mathbb{Z}), \\ HF^+(\#^n(S^1 \times S^2), s_0) &\cong \wedge^* H^1(\#^n(S^1 \times S^2); \mathbb{Z}) \otimes_{\mathbb{Z}} \left(\frac{\mathbb{Z}[U, U^{-1}]}{U \cdot \mathbb{Z}[U]} \right), \\ HF^-(\#^n(S^1 \times S^2), s_0) &\cong \wedge^* H^1(\#^n(S^1 \times S^2); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U] \\ HF^\infty(\#^n(S^1 \times S^2), s_0) &\cong \wedge^* H^1(\#^n(S^1 \times S^2); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}],\end{aligned}$$

where U^{-1} has grading 2.

In particular, there is a “top-dimensional” homology group of $\widehat{HF}(\#^n(S^1 \times S^2))$, which is isomorphic to \mathbb{Z} ; let $\widehat{\Theta}_n$ denote one of its generators. The image of this element in $HF^+(\#^n(S^1 \times S^2))$ is denoted Θ_n^+ . A generator for the top-dimensional homology group of $HF^-(\#^n(S^1 \times S^2), s_0) \subset HF^\infty(\#^n(S^1 \times S^2), s_0)$ is denoted Θ_n^- , while a generator for the top-dimensional homology group of $HF^{\leq 0}(\#^n(S^1 \times S^2))$ is denoted Θ_n . We drop the subscript n when it is clear from the context. Observe that all these elements are uniquely specified up to sign.

Definition 2.7. Let $(\Sigma, \alpha, \beta, z)$ be a Heegaard diagram for $\#^n(S^2 \times S^1)$ which is strongly admissible for s_0 , the Spin^c structure whose first Chern class vanishes. We call an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *maximal* if $s_z(\mathbf{x}) = s_0$ and the relative grading of \mathbf{x} (thought of as an element of $\widehat{CF}(\#^n(S^2 \times S^1))$) agrees with the relative grading of $\widehat{\Theta}_n \in \widehat{HF}(\#^n(S^2 \times S^1))$.

We can identify explicit representatives for the generators of $\widehat{HF}(\#^n(S^1 \times S^2), s_0)$, as follows.

Definition 2.8. Consider the pointed Heegaard diagram

$$(\Sigma, \{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}, z),$$

where Σ is an oriented two-manifold of genus n ; α_i is a small isotopic translate of β_i (via an isotopy supported in the complement of the basepoint z), meeting it in two

canceling transverse intersection points. This Heegaard diagram is called a *standard Heegaard diagram* for $\#^n(S^2 \times S^1)$.

For a standard Heegaard decomposition of $\#^n(S^2 \times S^1)$, the tori $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ meet in 2^n intersection points, corresponding to the generators of

$$\widehat{HF}(\#^n(S^1 \times S^2), s_0) \cong \wedge^* H^1(\#^n(S^1 \times S^2); \mathbb{Z}).$$

In particular, there is a unique maximal intersection point, inducing a representative of $\widehat{\Theta}_n$.

2.5. Proof of naturality

We now turn to a proof of Theorem 2.1.

Definition 2.9. Two Heegaard diagrams are said to be *strongly equivalent* if they differ by a sequence of isotopies and handleslides.

Lemma 2.10. Let $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ be two equivalent Heegaard diagrams, then both can be stabilized to give Heegaard diagrams $(\Sigma'_1, \alpha'_1, \beta'_1, z'_1)$ and $(\Sigma'_2, \alpha'_2, \beta'_2, z'_2)$ which are strongly equivalent.

Proof. According to Proposition 2.2 of [16], it follows that $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ can be connected by a sequence of Heegaard moves.

We claim that isotopies commute with stabilizations, in the following sense. Suppose that one can get from $(\Sigma, \alpha, \beta, z)$ to $(\Sigma', \alpha', \beta', z')$ by a sequence of isotopies and handleslides followed by stabilizations, then one can also get from the first Heegaard diagram by first a sequence of stabilizations, followed by isotopies and handleslides. This follows from the simple observation that it is equivalent to either

- isotope some α_i (resp. β_i) across a point w and then stabilize at w ,
- or to first stabilize at w and then handleslide α_i (resp. β_i) twice over the newly introduced α_g (resp. β_g).

In this way, stabilizations can be moved before isotopies and handleslides. They can also be moved before destabilizations: destabilizing a canceling pair of curves α_g and β_g , and then stabilizing at some point w is the same as first stabilizing at w and then destabilizing the original pair of curves.

Thus, we can commute all stabilizations to the beginning of the sequence of moves, and all destabilizations to the end. This is equivalent to the assertion made in the lemma. \square

Recall (cf. Section 4 of [16], see especially Theorem 6.1) that the Floer homology groups depend on a choice of complex structure j , and a generic path $J_s \subset \mathcal{U}$ of perturbations of the induced complex structure over $\text{Sym}^g(\Sigma)$, where \mathcal{U} is a contractible set of almost-complex structures, defined in [16].

Lemma 2.11. *Fix two different choices (j, J_s) and (j', J'_s) of complex structures and perturbations. Then, the induced isomorphism between Floer homologies*

$$HF_{J_s}^\circ(\alpha, \beta, \mathfrak{s}) \cong HF_{J'_s}^\circ(\alpha, \beta, \mathfrak{s})$$

are canonical, i.e. independent of paths connecting J_s and J'_s .

Proof. First, we suppose that $j = j'$, writing $J_s = J_s(0)$ and $J'_s = J_s(1)$. We can connect $J_s(0)$ and $J_s(1)$ by a one-parameter family $J_{s,t}$ as in the proof of Theorem 6.1 of [16] to construct a chain homotopy equivalence

$$\Phi_{J_{s,t}}^\infty: (CF^\infty(\alpha, \beta, \mathfrak{s}), \partial_{J_s(1)}^\infty) \longrightarrow (CF^\infty(\alpha, \beta, \mathfrak{s}), \partial_{J_s(0)}^\infty).$$

In fact, we claim that the induced map on homology by $\Phi_{J_{s,t}}^\infty$ is actually independent of the path of paths $J_{s,t}$ used in its definition (i.e. depending only on its endpoints). Since \mathcal{U} is contractible, any two such paths $J_{s,t}(0)$ and $J_{s,t}(1)$ can be connected by a homotopy $J_{s,t,\tau}$. We use this homotopy to define a map

$$H_{J_{s,t,\tau}}^\infty([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = -1\}} \#(\mathcal{M}_{J_{s,t,\tau}}(\phi)) \cdot [\mathbf{y}, i - n_z(\phi)].$$

which lowers degree by 1. Now, counting the ends of the moduli spaces $\mathcal{M}_{J_{s,t,\tau}}(\psi)$ with $\mu(\psi) = 0$, we see that $H_{J_{s,t,\tau}}^\infty$ provides a chain homotopy between $\Phi_{J_{s,t}(0)}^\infty$ and $\Phi_{J_{s,t}(1)}^\infty$.

Since $\Phi_{J_{s,t}}^\infty$ respects the filtration induced by n_z , the analogous results hold for CF^\pm as well.

The boundary maps are invariant under small perturbations (provided that the perturbations still give energy bounds), so—in view of the fact that the space of complex structures j over Σ is contractible—it follows also that the chain complexes are naturally identified as j is varied, as well. (Observe here that a codimension two subset of complex structures was removed from consideration in [16] to avoid difficulties occurring in moduli spaces with Maslov index two, but these do not arise in the definition of the boundary maps.) \square

Suppose now that $(\Sigma, \alpha_1, \beta_1, z)$ and $(\Sigma, \alpha_2, \beta_2, z)$ are strongly equivalent Heegaard diagrams both of which are admissible for \mathfrak{s} . Following Section 5 of [16], we can construct another pointed Heegaard diagram $(\Sigma, \alpha'_1, \beta'_1, z)$ so that:

- the curves α'_1 and β'_1 are connected to α_1 and β_1 , respectively, by special isotopies
- the Heegaard quadruple $(\Sigma, \alpha'_1, \beta'_1, \alpha_2, \beta_2, z)$ is strongly admissible, for the unique Spin^c structure over $X_{\alpha'_1, \beta'_1, \alpha_2, \beta_2}$ whose restriction to $(\Sigma, \alpha'_1, \beta_2, z) \cong Y$ is \mathfrak{s} and whose restriction to $(\Sigma, \alpha'_1, \alpha_2, z) \cong (\Sigma, \beta'_1, \beta_2, z) \cong \#^g(S^1 \times S^2)$ is \mathfrak{s}_0 .

We have an isomorphism induced by the special isotopies

$$\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}: CF^\circ(\alpha_1, \beta_1, s) \longrightarrow CF^\circ(\alpha'_1, \beta'_1, s)$$

defined by counting holomorphic disks with time-dependent boundary conditions (analogous to those maps $\Gamma_{\alpha', \alpha; \beta}$ discussed earlier, only now we use time-dependent boundary conditions on both boundaries of the strip). We define the strong equivalence map Φ to be the composite

$$CF^\circ(\alpha_1, \beta_1, s) \xrightarrow{\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}} CF^\circ(\alpha'_1, \beta'_1, s) \xrightarrow{\Theta_{\alpha_2, \alpha'_1} \otimes \cdot \otimes \Theta_{\beta'_1, \beta_2}} CF^\circ(\alpha_2, \beta_2, s),$$

where the last map is shorthand for the map

$$\zeta \mapsto F^\circ_{\alpha_2, \beta'_1, \beta_2}(F^\circ_{\alpha_2, \alpha'_1, \beta'_1}(\Theta_{\alpha_2, \alpha'_1} \otimes \zeta, s) \otimes \Theta_{\beta'_1, \beta_2}, s),$$

where

$$F^\circ_{\alpha_2, \alpha'_1, \beta'_1}: CF^{\leq 0}(\alpha_2, \alpha'_1, s_0) \otimes CF^\circ(\alpha'_1, \beta'_1, s) \longrightarrow CF^\circ(\alpha_2, \beta'_1, s)$$

is a version of the map associated to holomorphic triangles, where we use $CF^{\leq 0}$ on the first factor (rather than the second, as usual). Of course, the strong equivalence map Φ depends on the auxiliary Heegaard diagram $(\Sigma, \alpha'_1, \beta'_1, z)$, as well as the isotopies connecting α_1 to α'_1 and β_1 to β'_1 .

To show that the map induced on homology is independent of the various choices, we use some naturality properties of the maps induced by isotopies.

Lemma 2.12. *The maps Γ are natural under composition of isotopies. Specifically, if we fix isotopies from α to α' , and isotopies from α' to α'' (which never cross the reference point z in a Heegaard diagram $(\Sigma, \alpha, \beta, z)$), then the maps on homology $\Gamma_{\alpha'', \alpha; \beta}$ induced by the composite of the two isotopies equals the composition $\Gamma_{\alpha'', \alpha'; \beta} \circ \Gamma_{\alpha', \alpha; \beta}$. Similarly, if we have isotopies of α to α' and β to β' , then (on homology),*

$$\Gamma_{\alpha', \alpha; \beta, \beta'} = \Gamma_{\alpha', \alpha; \beta'} \circ \Gamma_{\alpha; \beta, \beta'} = \Gamma_{\alpha'; \beta, \beta'} \circ \Gamma_{\alpha', \alpha; \beta}.$$

Proof. This result is a variation on the argument that $\Gamma_{\alpha; \beta, \beta'}$ is an isomorphism on homology, as described in Theorem 7.3 of [16] (which in turn is a variation on familiar arguments from Floer theory). We focus on the first statement: suppose that Ψ'_t is an isotopy from α to α' and Ψ''_t is an isotopy from α' to α'' . We consider holomorphic disks with time-dependent boundary conditions, given by juxtaposing Ψ_t and Ψ'_t , with

a “long gap” in between: i.e. we define a one-parameter family of isotopies

$$\Psi_{\tau,t} = \begin{cases} \Psi'_{t-\tau} & \text{if } t \geq 0, \\ \Psi''_{t+\tau} & \text{if } t \leq 0. \end{cases}$$

We can count points in Ψ_τ -time-dependent moduli spaces. Taking the limit as $\tau \mapsto \infty$, this can be used to construct the chain homotopy between $\Gamma_{\alpha'',\alpha';\beta} \circ \Gamma_{\alpha';\alpha;\beta}$ and $\Gamma_{\alpha'',\alpha;\beta}$ as required. The other assertions follow similarly. \square

Lemma 2.13. *Up to sign, the map on homology induced by a strong equivalence*

$$\Phi: HF^\circ(\alpha_1, \beta_1, \mathfrak{s}) \longrightarrow HF^\circ(\alpha_2, \beta_2, \mathfrak{s})$$

is well-defined.

Proof. We must show that Φ is independent of the intermediate Heegaard diagram and the isotopy. This follows from the following commutative diagram:

$$\begin{array}{ccccc} HF^\circ(\alpha_1, \beta_1, \mathfrak{s}) & \xrightarrow{\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}} & HF^\circ(\alpha'_1, \beta'_1, \mathfrak{s}) & \xrightarrow{\Theta_{\alpha_2, \alpha'_1} \otimes \cdot \otimes \Theta_{\beta'_1, \beta_2}} & HF^\circ(\alpha_2, \beta_2, \mathfrak{s}) \\ \downarrow & & \Gamma_{\alpha''_1, \alpha'_1; \beta'_1, \beta''_1} \downarrow & & \downarrow \\ HF^\circ(\alpha_1, \beta_1, \mathfrak{s}) & \xrightarrow{\Gamma_{\alpha''_1, \alpha_1; \beta_1, \beta''_1}} & HF^\circ(\alpha''_1, \beta''_1, \mathfrak{s}) & \xrightarrow{\Theta_{\alpha_2, \alpha''_1} \otimes \cdot \otimes \Theta_{\beta''_1, \beta_2}} & HF^\circ(\alpha_2, \beta_2, \mathfrak{s}) \end{array}.$$

Commutativity of the left-hand square follows from the naturality of the Γ -maps under juxtaposition of isotopies (Lemma 2.12 above). Commutativity of the right-hand square up to sign follows from the isotopy invariance of the triangle construction (cf. Lemma 8.14 of [16]), together with the observation that

$$\Gamma_{\beta'_1, \beta''_1; \beta_2}(\Theta_{\beta''_1, \beta_2}) = \pm \Theta_{\beta'_1, \beta_2} \quad \text{and} \quad \Gamma_{\alpha_2; \alpha'_1, \alpha''_1}(\Theta_{\alpha_2, \alpha'_1}) = \pm \Theta_{\alpha_2, \alpha'_1} \quad (6)$$

since both sides of both equations represent generators of a groups which are isomorphic to \mathbb{Z} . For the reader's convenience, we break the verification of the commutativity of this right-hand square down into the following steps, each of which is an application of either the isotopy invariance of the triangle construction (the last two commutative diagrams in Theorem 2.3), the naturality of the isotopy maps (Lemma 2.12), or Eqs. (6):

$$\begin{aligned} & F_{\alpha_2, \beta''_1, \beta_2}^\circ(F_{\alpha_2, \alpha''_1, \beta''_1}^\circ(\Theta_{\alpha_2, \alpha''_1} \otimes \Gamma_{\alpha'_1, \alpha'_1; \beta'_1, \beta''_1}(\xi)) \otimes \Theta_{\beta''_1, \beta_2}) \\ &= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ(\Gamma_{\alpha_2; \beta''_1, \beta'_1}(F_{\alpha_2, \alpha''_1, \beta''_1}^\circ(\Theta_{\alpha_2, \alpha''_1} \otimes \Gamma_{\alpha'_1, \alpha'_1; \beta'_1, \beta''_1}(\xi))) \otimes \Gamma_{\beta'_1, \beta''_1; \beta_2}(\Theta_{\beta''_1, \beta_2})) \\ &= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ(F_{\alpha_2, \alpha''_1, \beta'_1}^\circ(\Theta_{\alpha_2, \alpha''_1} \otimes \Gamma_{\alpha'_1, \beta''_1, \beta'_1} \circ \Gamma_{\alpha'_1, \alpha'_1; \beta'_1, \beta''_1}(\xi))) \otimes \Gamma_{\beta'_1, \beta''_1; \beta_2}(\Theta_{\beta''_1, \beta_2})) \end{aligned}$$

$$\begin{aligned}
&= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ (F_{\alpha_2, \alpha'_1, \beta'_1}^\circ (\Theta_{\alpha_2, \alpha'_1} \otimes \Gamma_{\alpha'_1, \alpha'_1; \beta'_1}(\xi)) \otimes \Gamma_{\beta'_1, \beta'_1; \beta_2}(\Theta_{\beta'_1, \beta_2})) \\
&= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ (F_{\alpha_2, \alpha'_1, \beta'_1}^\circ (\Theta_{\alpha_2, \alpha'_1} \otimes \Gamma_{\alpha'_1, \alpha'_1; \beta'_1}(\xi)) \otimes \Theta_{\beta'_1, \beta_2}) \\
&= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ (F_{\alpha_2, \alpha'_1, \beta'_1}^\circ (\Gamma_{\alpha_2, \alpha'_1, \alpha'_1}(\Theta_{\alpha_2, \alpha'_1}) \otimes \xi) \otimes \Theta_{\beta'_1, \beta_2}) \\
&= \pm F_{\alpha_2, \beta'_1, \beta_2}^\circ (F_{\alpha_2, \alpha'_1, \beta'_1}^\circ (\Theta_{\alpha_2, \alpha'_1} \otimes \xi) \otimes \Theta_{\beta'_1, \beta_2}).
\end{aligned}$$

(Here, when we go from the second to the third equation, we use a variant of the first commutative diagram in Theorem 2.3, only varying γ rather than α .) \square

Suppose now that $(\Sigma^s, \alpha^s, \beta^s, z)$ is obtained from $(\Sigma, \alpha, \beta, z)$ by a stabilization; i.e. we let E be an oriented two-manifold of genus one with a pair α_{g+1} and β_{g+1} of embedded curves meeting transversally in a single intersection point c , and let $\Sigma^s = \Sigma \# E$, $\alpha^s = \alpha \cup \{\alpha_{g+1}\}$, and $\beta^s = \beta \cup \{\beta_{g+1}\}$. Then we have defined an isomorphism belonging to the stabilization (cf. [16], Section 10)

$$\sigma: HF^\circ(\alpha, \beta, s) \longrightarrow HF^\circ(\alpha^s, \beta^s, s),$$

which (for appropriately chosen complex structures and perturbations) is given by the map which associates to each intersection point $\mathbf{x} \in \text{Sym}^g(\Sigma)$, the stabilized point $\mathbf{x} \times \{c\} \in \text{Sym}^{g+1}(\Sigma \# E)$.

If we stabilize two strongly equivalent Heegaard diagrams, the resulting diagrams are strongly equivalent, as well. We wish to show that the isomorphism induced by strong equivalence commutes with the isomorphism induced by stabilization. To this end, we shall employ the gluing theorem for holomorphic triangles, Theorem 9.4 of [15], which we state here for convenience:

Theorem 2.14. *Fix a pair of Heegaard diagrams*

$$(\Sigma, \alpha, \beta, \gamma, z) \quad \text{and} \quad (E, \alpha_0, \beta_0, \gamma_0, z_0),$$

where E is a Riemann surface of genus one. Consider the connected sum $\Sigma \# E$, where the connected sum points are near the distinguished points z and z_0 , respectively. Fix intersection points $\mathbf{x}, \mathbf{y}, \mathbf{w}$ for the first diagram and a class $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$, and intersection points x_0, y_0 , and w_0 for the second, with a triangle $\psi_0 \in \pi_2(x_0, y_0, w_0)$ with $\mu(\psi) = \mu(\psi_0) = 0$. Suppose moreover that $n_{z_0}(\psi_0) = 0$. Then, for a suitable choice of complex structures and perturbations, we have a diffeomorphism of moduli spaces:

$$\mathcal{M}(\psi') \cong \mathcal{M}(\psi) \times \mathcal{M}(\psi_0),$$

where $\psi' \in \pi_2(\mathbf{x} \times x_0, \mathbf{y} \times y_0, \mathbf{w} \times w_0)$ is the triangle for $\Sigma \# E$ whose domain on the Σ -side agrees with $\mathcal{D}(\psi)$, and whose domain on the E -side agrees with $\mathcal{D}(\psi_0) + n_z(\psi)[E]$.

Lemma 2.15. *The isomorphism induced by stabilization commutes with that induced by equivalence.*

Proof. Let α'_{g+1} and β'_{g+1} be small Hamiltonian isotopic translates of α_{g+1} and β_{g+1} in E . Also, let α'_1 and β'_1 be isotopic translates of the α_1 and β_1 required in the definition of Φ , chosen so that the isotopies are supported away from the stabilization point. We claim then that the following diagram commutes:

$$\begin{array}{ccc} HF^\circ(\alpha_1, \beta_1) & \xrightarrow{\Gamma_{\alpha'_1, \alpha_1; \beta_1}} & HF^\circ(\alpha'_1, \beta'_1) \\ \sigma_1 \downarrow & & \downarrow \sigma'_1 \\ HF^\circ(\alpha_1^s, \beta_1^s) & \xrightarrow{\Gamma_{\alpha_1^s, \alpha'_1; \beta_1^s, \beta'_1}} & HF^\circ(\alpha_1^s, \beta_1^s) \end{array}$$

where here α_1^s , β_1^s , $\alpha_1'^s$, and $\beta_1'^s$ denote the stabilizations $\alpha_1 \cup \{\alpha_{g+1}\}$, $\beta_1 \cup \{\beta_{g+1}\}$, $\alpha'_1 \cup \{\alpha'_{g+1}\}$, and $\beta'_1 \cup \{\beta'_{g+1}\}$, respectively. This follows from the analogue of the gluing theorem for stabilization invariance ([16], Theorem 10.2), only using time-dependent flow-lines.

Moreover, we have commutativity of

$$\begin{array}{ccc} HF^\circ(\alpha'_1, \beta'_1) & \xrightarrow{\Theta_{\alpha_2, \alpha'_1} \otimes \cdots \otimes \Theta_{\beta'_1, \beta_2}} & HF^\circ(\alpha_2, \beta_2) \\ \sigma'_1 \downarrow & & \downarrow \sigma_2 \\ HF^\circ(\alpha_1'^s, \beta_1'^s) & \xrightarrow{\Theta_{\alpha_2^s, \alpha_1'^s} \otimes \cdots \otimes \Theta_{\beta_1'^s, \beta_2^s}} & HF^\circ(\alpha_2^s, \beta_2^s) \end{array}$$

This follows from an application of Theorem 2.14 (see also Lemma 4.7 below).

Together, these two commutative squares show that stabilization commutes with the map associated to strong equivalence. \square

Proof of Theorem 2.1. Suppose that $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ represent the same three-manifold Y , and both are admissible for $\mathfrak{s} \in \text{Spin}^c(Y)$. Then, we connect them as in Lemma 2.10 and define Ψ° to be the composite of the stabilization isomorphism composed with the equivalence isomorphism, composed with the inverse of the final stabilization isomorphism.

We must argue that the map Ψ° is independent of the choice of common stabilization $\Sigma'_1 = \Sigma'_2$. Now, if Σ'_1 is another intermediate choice, then there is a third stabilization Σ'''_1 which is obtained by stabilizing both Σ'_1 and Σ''_1 . The map induced by factoring through Σ'''_1 agrees by that induced through Σ'_1 (and also Σ''_1) in view of Lemma 2.15.

Observe that the equivalence isomorphisms Ψ° are equivariant under the action of $\Lambda^* H_1(Y, \mathfrak{s})/\text{Tors}$, as was verified in Section 4 of [16].

The fact that Ψ° induces a map of the long exact sequences for \widehat{HF} (Eq. (1)) is a formal consequence of the fact that the isomorphism induced by isotopies carries

$\widehat{CF}(\alpha, \beta, t) \subset CF^+(\alpha, \beta, t)$ to the corresponding \widehat{CF} for the isotopic Heegaard diagram, together with the fact that the map $F^+(\cdot, s)$ induced by counting holomorphic triangles is U -equivariant.

2.6. Twisted coefficients and naturality

Let $H = H^1(Y; \mathbb{Z})$. There is a Floer homology with twisted coefficients, denoted $\underline{HF}^\circ(Y, t)$, which is a module over the ring $\mathbb{Z}[H]$. We recall the construction briefly.

Recall that there is always a natural map from $\pi_2(\mathbf{x}, \mathbf{x})$ to H , which is obtained as follows. Each $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ naturally gives rise to an associated two-chain in Σ whose boundary is a collection of circles among the α and β . We can then close off the two-chain to give a closed two-cycle in Y by gluing on copies of the attaching disks for the handlebodies in the Heegaard diagram for Y . The Poincaré dual of this two-cycle is the associated element of $H^1(Y; \mathbb{Z})$.

Fix a t -admissible Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y , and an *additive assignment* in the sense of Definition 2.12 of [16], i.e. letting $\mathfrak{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be the set of intersection points \mathbf{x} so that $s_z(\mathbf{x}) = t$, we fix a collection of maps:

$$A = \{A_{\mathbf{x}, \mathbf{y}}: \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow H^1(Y; \mathbb{Z})\}_{\mathbf{x}, \mathbf{y} \in \mathfrak{S}},$$

so that:

- when $\mathbf{x} = \mathbf{y}$, $A_{\mathbf{x}, \mathbf{x}}$ is the canonical map from $\pi_2(\mathbf{x}, \mathbf{x})$ onto $H^1(Y; \mathbb{Z})$ defined above
- A is compatible with splicing in the sense that if $\mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathfrak{S}$, then for each $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{w})$, we have that $A(\phi_1 * \phi_2) = A(\phi_1) + A(\phi_2)$.

We write elements of the group-ring $\mathbb{Z}[H]$ as finite sums $\sum_{h \in H} n_h \cdot e^h$ (with $n_h \in \mathbb{Z}$). Consider the chain complex $\underline{CF}^\infty(Y, t, A)$ which is freely generated as a $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module by elements $[\mathbf{x}, i]$ where $\mathbf{x} \in \mathfrak{S}$ and $i \in \mathbb{Z}$. The boundary map is defined by

$$\partial^\infty[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathfrak{S}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} (\#\widehat{\mathcal{M}}(\phi)) \cdot e^{A(\phi)} \otimes [\mathbf{y}, i - n_z(\phi)].$$

The homology groups of this complex are the completely twisted homology groups $\underline{HF}^\circ(Y, t)$. More generally, if M is any module over $\mathbb{Z}[H]$, then $\underline{HF}(Y, t, M)$ is defined to be the homology of

$$\underline{CF}^\circ(Y, t, M, A) \cong \underline{CF}^\circ(Y, t, A) \otimes_{\mathbb{Z}[H]} M.$$

Observe that the homology is independent of the additive assignment A . Specifically, if we have two additive assignments A and A' , then we can define an isomorphism of chain complexes

$$\Phi: \underline{CF}^\infty(Y, t, M, A) \longrightarrow \underline{CF}^\infty(Y, t, M, A')$$

as follows. Fix a point $\mathbf{x}_0 \in \mathfrak{S}$, and let

$$\Phi(m \otimes [\mathbf{x}, i]) = (m \cdot e^{A'(\phi) - A(\phi)}) \otimes [\mathbf{x}, i],$$

where ϕ is any element of $\pi_2(\mathbf{x}_0, \mathbf{x})$. It is easy to see that Φ is an isomorphism of chain complexes (over $\mathbb{Z}[H]$); but the actual isomorphism depends up to translation by an element of H on the initial point $\mathbf{x}_0 \in \mathfrak{S}$. Indeed, the homology groups are topological invariants, according to the following:

Theorem 2.16. *Fix a module M for $\mathbb{Z}[H]$, and suppose that $(\Sigma, \alpha, \beta, z)$ and $(\Sigma', \alpha', \beta', z')$ are equivalent Heegaard diagrams which are admissible for the Spin^c structure \mathfrak{t} , then there are induced isomorphisms of the corresponding long exact sequences:*

$$\begin{array}{ccccc} \underline{HF}^-(\alpha, \beta, \mathfrak{t}, M) & \xrightarrow{i} & \underline{HF}^\infty(\alpha, \beta, \mathfrak{t}, M) & \xrightarrow{\pi} & \underline{HF}^+(\alpha, \beta, \mathfrak{t}, M) \\ \underline{\Psi}^- \downarrow & & \underline{\Psi}^\infty \downarrow & & \underline{\Psi}^+ \downarrow \\ \underline{HF}^-(\alpha', \beta', \mathfrak{t}, M) & \xrightarrow{i'} & \underline{HF}^\infty(\alpha', \beta', \mathfrak{t}, M) & \xrightarrow{\pi'} & \underline{HF}^+(\alpha', \beta', \mathfrak{t}, M) \end{array}$$

(i.e. where each square commutes), where the vertical maps commute with the actions of $\mathbb{Z}[U]$. Moreover, the maps $\underline{\Psi}^-$, $\underline{\Psi}^\infty$, and $\underline{\Psi}^+$ are uniquely determined up to multiplication by ± 1 and translation in H . There is a similar canonically induced map of the long exact sequence for \widehat{HF} .

The above theorem is proved the same way as Theorem 2.1 is proved; with suitable modifications made to the holomorphic triangle construction which allow for twisted coefficients, and induced modules, as we describe in the next subsection.

Of course, the chain complex $\underline{CF}^\circ(Y, \mathfrak{t}, M)$ is obtained from the chain complex in the totally twisted case $\underline{CF}^\circ(Y, \mathfrak{t})$ by a change of coefficients; thus, the corresponding homology groups are related by a universal coefficients spectral sequence (cf. [1]). In particular, when M is the trivial $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module \mathbb{Z} , the M -twisted chain complex is the same as the untwisted chain complex stated earlier.

2.7. Holomorphic triangles and twisted coefficients

We set this up with slightly less generality than in [15], but with sufficient generality for the applications in the present paper.

Let W be a cobordism from Y_1 to Y_2 , and fix a module M for $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$. The group

$$K(W) = \text{Ker} \left(H^2(W, \partial W; \mathbb{Z}) \longrightarrow H^2(W; \mathbb{Z}) \right)$$

can be used to induce a module $M(W)$ for $\mathbb{Z}[H^1(Y_2; \mathbb{Z})]$, defined by

$$M(W) = M \otimes_{\mathbb{Z}[H^1(Y_1; \mathbb{Z})]} \mathbb{Z}[K(W)].$$

Suppose now that $(\Sigma, \alpha, \beta, \gamma, z)$ is a Heegaard triple, and suppose that $(\Sigma, \beta, \gamma, z)$ represents $\#^n(S^2 \times S^1)$, so that when we fill it in with $\#^n(D^3 \times S^1)$, we obtain the cobordism W above. Fix a Spin^c structure \mathfrak{s} over W , and let \mathfrak{S} denote the set of homotopy classes of triangles which induce the fixed Spin^c structure \mathfrak{s} over $X_{\alpha, \beta, \gamma}$.

Now, two additive assignments $A_{\alpha, \beta}$, $A_{\beta, \gamma}$, $A_{\alpha, \gamma}$ for the corresponding Heegaard diagrams on the boundary and a single choice of $\psi_0 \in \mathfrak{S}$ give rise to a map

$$A_W: \mathfrak{S} \longrightarrow K(W),$$

defined as follows. If $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \cap \mathfrak{S}$, then we can find paths $\phi_{\alpha, \beta} \in \pi_2(\mathbf{x}_0, \mathbf{x})$, $\phi_{\beta, \gamma} \in \pi_2(\mathbf{y}_0, \mathbf{y})$ and $\phi_{\alpha, \gamma} \in \pi_2(\mathbf{w}_0, \mathbf{w})$ with the property that

$$\psi = \psi_0 + \phi_{\alpha, \beta} + \phi_{\beta, \gamma} + \phi_{\alpha, \gamma}.$$

Then, we define

$$A_W(\psi) = \delta(A(\phi_{\alpha, \beta}) \oplus A(\phi_{\alpha, \gamma})),$$

where δ denotes the coboundary map $\delta: H^1(\partial W; \mathbb{Z}) \longrightarrow K(W)$. It is easy to see that this is independent of the choices of $\phi_{\alpha, \beta}$, $\phi_{\beta, \gamma}$ and $\phi_{\alpha, \gamma}$ as above.

We now claim that the holomorphic triangle construction gives a map

$$\underline{F}_{\alpha, \beta, \gamma}^\circ: \underline{HF}^\circ(\alpha, \beta, \mathfrak{s}|Y_1, M) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_0) \longrightarrow \underline{HF}^\circ(\alpha, \gamma, \mathfrak{s}|Y_2, M(W)),$$

to be the map on homology induced by the chain map \underline{f}° defined by:

$$\begin{aligned} \underline{f}_{\alpha, \beta, \gamma}^\circ(m \otimes [\mathbf{x}, i] \otimes [\mathbf{y}, j]; \mathfrak{s}) \\ = \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \mid \mathfrak{s}_z(\psi) = \mathfrak{s}, \mu(\psi) = 0\}} \left(\# \mathcal{M}(\psi) \right) m \otimes e^{A_W(\psi)} \otimes [\mathbf{w}, i + j - n_z(\psi)]. \end{aligned}$$

This map commutes with the obvious $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$ -actions.

Although this construction depends on an initial triangle $\psi_0 \in \mathfrak{S}$, the following is clear: If $\psi_1 \in \mathfrak{S}$ is another such choice, then we can write $\psi_1 = \psi_0 + \phi_{\alpha, \beta} + \phi_{\beta, \gamma} + \phi_{\alpha, \gamma}$ with $A_{\beta, \gamma}(\phi_{\beta, \gamma}) = 0$. Then, if \underline{F}_0° and \underline{F}_1° are the two induced maps, then we have

$$e^{A_{\alpha, \gamma}(\phi_{\alpha, \gamma})} \cdot \underline{F}_0^\circ = \underline{F}_1^\circ \cdot e^{A_{\alpha, \beta}(\phi_{\alpha, \beta})}.$$

Note that if $\gamma: M \longrightarrow M'$ is a map of $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -modules, then there is an induced map

$$H(\gamma): \underline{HF}^\circ(Y, M) \longrightarrow \underline{HF}^\circ(Y, M')$$

defined in the obvious way.

The map \underline{F}° is a refinement of the map

$$F^\circ: HF^\circ(\alpha, \beta, \mathfrak{s}|Y_1) \otimes HF^{\leq 0}(\beta, \gamma, \mathfrak{s}_0) \longrightarrow HF^\circ(\alpha, \gamma, \mathfrak{s}|Y_2)$$

defined earlier, in the following sense. Observe first that if \mathbb{Z} is a trivial $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module, then $\underline{HF}^\circ(Y, \mathfrak{s}, \mathbb{Z}) = HF^\circ(Y, \mathfrak{s})$. Moreover, if W is a cobordism from Y_1 to Y_2 , and K denotes the kernel of $H^2(W, Y_2) \longrightarrow H^2(W)$, then the trivial $H^1(Y_1; \mathbb{Z})$ -module \mathbb{Z} induces the $H^1(Y_2; \mathbb{Z})$ -module $\mathbb{Z}[K]$. Now, letting $\varepsilon: \mathbb{Z}[K] \longrightarrow \mathbb{Z}$ be the natural map (to the trivial $H^1(Y_2; \mathbb{Z})$ -module) and $H(\varepsilon): \underline{HF}^\circ(Y, \mathbb{Z}[K]) \longrightarrow HF^\circ(Y)$ be its induced map on homology, we have that

$$F^\circ(\zeta \otimes \eta, \mathfrak{s}) = H(\varepsilon) \circ \underline{F}^\circ(\zeta \otimes \eta, \mathfrak{s}). \quad (7)$$

3. Cobordisms

In Section 4, we will use holomorphic triangles to construct invariants (which are maps on the HF°) induced by one-, two-, and three-handle addition to a given three-manifold. By composing these invariants, we define the invariant for cobordisms. We state these properties in the present section, returning to verifications of the properties in Sections 4, 5, and 6.

Theorem 3.1. *Let W be an oriented, smooth, connected, four-dimensional cobordism with $\partial W = -Y_1 \cup Y_2$. Fix a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(W)$, and let t_i denote its restriction to Y_i . Then, by composing the maps associated to a handle decomposition of W , we obtain a map*

$$F_{W, \mathfrak{s}}^\circ: HF^\circ(Y_1, t_1) \longrightarrow HF^\circ(Y_2, t_2)$$

which is a smooth oriented four-manifold invariant, uniquely defined up to sign. Indeed, if $HF^\circ(Y_2, t_2)$ has no torsion (over \mathbb{Z}), then $F_{W, \mathfrak{s}}^\circ$ can be extended to a map

$$F_{W, \mathfrak{s}}^\circ: HF^\circ(Y_1, t_1) \otimes \Lambda^*(H_1(W, \mathbb{Z})/\text{Tors}) \longrightarrow HF^\circ(Y_2, t_2),$$

which is also a four-manifold invariant. More precisely, if $\phi: W \longrightarrow W'$ is an orientation-preserving diffeomorphism, then we have a commutative diagram:

$$\begin{array}{ccc} HF^\circ(Y_1, t_1) \otimes \Lambda^*(H_1(W; \mathbb{Z})/\text{Tors}) & \xrightarrow{F_{W, \mathfrak{s}}} & HF^\circ(Y_2, t_2) \\ (\phi|_{Y_1})_* \otimes \phi_* \downarrow & & \downarrow (\phi|_{Y_2})_* \\ HF^\circ(Y'_1, t'_1) \otimes \Lambda^*(H_1(W'; \mathbb{Z})/\text{Tors}) & \xrightarrow{F_{W', \mathfrak{s}'}} & HF^\circ(Y_2, t'_2) \end{array}$$

where $\mathfrak{s} = \phi^(\mathfrak{s}')$.*

Remark 3.2. When we write

$$F_{W,s}^\circ: HF^\circ(Y_1, t_1) \longrightarrow HF^\circ(Y_2, t_2),$$

we mean a map between the long exact sequences relating \widehat{HF} , HF^- , HF^∞ , and HF^+ . Specifically, the above theorem is saying that there is a collection of maps $\widehat{F}_{W,s}^-$, $F_{W,s}^-$, $F_{W,s}^\infty$ and $F_{W,s}^+$ for which the following diagrams commute:

$$\begin{array}{ccccccc} \dots & \longrightarrow & HF^-(Y_1, t_1) & \xrightarrow{l} & HF^\infty(Y_1, t_1) & \xrightarrow{\pi} & HF^+(Y_1, t_1) \longrightarrow \dots \\ & & \downarrow F_{W,s}^- & & \downarrow F_{W,s}^\infty & & \downarrow F_{W,s}^+ \\ \dots & \longrightarrow & HF^-(Y_2, t_2) & \xrightarrow{l} & HF^\infty(Y_2, t_2) & \xrightarrow{\pi} & HF^+(Y_2, t_2) \longrightarrow \dots \end{array}$$

and also

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widehat{HF}(Y_1, t_1) & \xrightarrow{\widehat{l}} & HF^+(Y_1, t_1) & \xrightarrow{U} & HF^+(Y_1, t_1) \longrightarrow \dots \\ & & \downarrow \widehat{F}_{W,s} & & \downarrow F_{W,s}^+ & & \downarrow F_{W,s}^+ \\ \dots & \longrightarrow & \widehat{HF}(Y_2, t_2) & \xrightarrow{\widehat{l}} & HF^+(Y_2, t_2) & \xrightarrow{U} & HF^+(Y_2, t_2) \longrightarrow \dots \end{array}$$

This invariant enjoys a number of fundamental properties.

Theorem 3.3 (Finiteness). *Let W be a cobordism from Y_1 to Y_2 , and fix Spin^c structures t_1 and t_2 over W . Then, for each $\xi \in HF^+(Y_1, t_1)$, there are only finitely many $s \in \text{Spin}^c(W)$ for which*

$$F_{W,s}^+(\xi) \neq 0.$$

Moreover, for each integer d , and each element $\eta \in HF^-(Y_1, t_1)$, there are only finitely many Spin^c structures $s \in \text{Spin}^c(W)$ for which

$$F_{W,s}^-(\eta) \notin U^d HF^-(Y_2, t_2).$$

Theorem 3.4 (Composition law). *Let W_1 and W_2 be a pair of connected cobordisms with $\partial W_1 = -Y_1 \cup Y_2$, $\partial W_2 = -Y_2 \cup Y_3$, and let $W = W_1 \cup_{Y_2} W_2$ be their composite. Fix Spin^c structures $s_i \in \text{Spin}^c(W_i)$ for $i = 1, 2$ with $s_1|_{Y_2} = s_2|_{Y_2}$. Then, $F_{W_2, s_2}^\circ \circ F_{W_1, s_1}^\circ$ can be written as a sum:*

$$F_{W_2, s_2}^\circ \circ F_{W_1, s_1}^\circ = \sum_{\{s \in \text{Spin}^c(W) \mid s|_{W_1} = s_1, s|_{W_2} = s_2\}} \pm F_{W, s}^\circ.$$

There is a natural bilinear pairing (cf. Section 5)

$$\langle \cdot, \cdot \rangle: CF^\infty(Y, \mathfrak{s}) \otimes CF^\infty(-Y, \mathfrak{s}) \longrightarrow \mathbb{Z}$$

(cf. Section 5) which descends to give pairings on $HF^\infty(Y, \mathfrak{s}) \otimes HF^\infty(-Y, \mathfrak{s})$, $HF^+(Y, \mathfrak{s}) \otimes HF^+(-Y, \mathfrak{s})$, and $HF_{\text{red}}^+(Y, \mathfrak{s}) \otimes HF_{\text{red}}^+(-Y, \mathfrak{s})$.

Theorem 3.5 (Duality). *Let W be a cobordism from Y_1 to Y_2 . Then, the map induced by W , thought of as a cobordism from Y_1 to Y_2 , is dual to the map induced by W , thought of as a cobordism from $-Y_2$ to $-Y_1$; i.e. for each $\xi \in HF^+(Y_1, \mathfrak{t}_1)$, $\eta \in HF^+(-Y_2, \mathfrak{t}_2)$, we have that*

$$\langle F_{W, \mathfrak{s}}^+(\xi), \eta \rangle_{HF^+(Y_2, \mathfrak{t}_2) \otimes HF^+(-Y_2, \mathfrak{t}_2)} = \langle \xi, F_{W, \mathfrak{s}}^-(\eta) \rangle_{HF^+(Y_1, \mathfrak{t}_1) \otimes HF^+(-Y_1, \mathfrak{t}_1)}.$$

In general, there is a $\mathbb{Z}/2\mathbb{Z}$ action on the set of Spin^c structures over a given manifold. In three dimensions, thinking of Spin^c structures as equivalence classes of nowhere-vanishing vector fields v , this action is induced by the map $v \mapsto -v$. In four dimensions, thinking of Spin^c structures as equivalence classes of almost-complex structures J defined away from a finite collection of points, the map is induced by the map $J \mapsto -J$.

In [16], we defined an isomorphism of chain complexes, which induces an identification $HF^\circ(Y, \mathfrak{s})$ with $HF^\circ(Y, \bar{\mathfrak{s}})$. We denote the isomorphism by $\mathfrak{I}_Y: HF^\circ(Y, \mathfrak{s}) \longrightarrow HF^\circ(Y, \bar{\mathfrak{s}})$.

Theorem 3.6 (Conjugation invariance). *Let W be a cobordism as above. Then, $F_{W, \mathfrak{s}}^\circ = \mathfrak{I}_{Y_2} \circ F_{W, \bar{\mathfrak{s}}}^\circ \circ \mathfrak{I}_{Y_1}$.*

Proofs of duality and conjugation invariance will be given in Section 5.

Theorem 3.7 (Blow-up formula). *Let W be the cobordism $([0, 1] \times Y) \# \overline{\mathbb{CP}}^2$ (internal connected sum). Then, for each Spin^c structure \mathfrak{t} over Y , we have that*

$$F_{W, \mathfrak{s}}^\circ: HF^\circ(Y, \mathfrak{t}) \longrightarrow HF^\circ(Y, \mathfrak{t})$$

is multiplication by $U^{\frac{\ell(\ell+1)}{2}}$ where $\mathfrak{s} \in \text{Spin}^c W$ is characterized by $c_1(\mathfrak{s})|_{\{0\} \times Y} = \mathfrak{t}$ and $\langle c_1(\mathfrak{s}), E \rangle = \pm(2\ell + 1)$, for $\ell \geq 0$.

3.1. Twisted analogues

We give the following twisted versions of Theorems 3.1 and 3.4.

Fix first a module M over $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$, and a cobordism W from Y_1 to Y_2 as in Theorem 3.1. Let $M(W)$ be the induced module $M(W)$ over $\mathbb{Z}[H^1(Y_2; \mathbb{Z})]$ induced by the cobordism as in Section 2.7.

We have the following:

Theorem 3.8. *Fix a Spin^c structure \mathfrak{s} over W . Then, there is an associated map*

$$F_{W,\mathfrak{s},M}^\circ: \underline{HF}^\circ(Y_1, \mathfrak{s}|_{Y_1}, M) \longrightarrow \underline{HF}^\circ(Y_2, \mathfrak{s}|_{Y_2}, M(W))$$

which is uniquely defined up to multiplication by ± 1 , left-translation by an element of $H^1(Y_1; \mathbb{Z})$, and right translation by an element of $H^1(Y_2; \mathbb{Z})$.

We let $[F_{W,\mathfrak{s}}^\circ]$ be the $H^1(Y_1; \mathbb{Z}) \oplus H^1(Y_2; \mathbb{Z})$ -orbit of the map constructed in Theorem 3.8.

We state a refined version of the composition law. Let W_1 be a cobordism from Y_1 to Y_2 and W_2 be a cobordism from Y_2 to Y_3 , and let $W = W_1 \cup_{Y_2} W_2$ be their composite. Then there is a natural inclusion

$$i_M: M(W) \longrightarrow M(W_1)(W_2),$$

which has a canonically defined right-inverse

$$\Pi_M: M(W_1)(W_2) \longrightarrow M(W).$$

To construct this, note that there is an $H^1(Y_2; \mathbb{Z})$ -equivariant inclusion

$$i: K(W) \longrightarrow \frac{K(W_1) \oplus K(W_2)}{H^1(Y_2; \mathbb{Z})},$$

as can be seen by inspecting the following commutative diagram:

$$\begin{array}{ccccccc} H^1(Y_2) & \xrightarrow{\delta} & H^1(W_1, \partial W_1) \oplus H^1(W_2, \partial W_2) & \longrightarrow & H^2(W, \partial W) & \longrightarrow & H^2(Y_2) \\ & & & & \downarrow & & \uparrow \\ & & & & H^2(W) & \longrightarrow & H^2(W_1) \oplus H^2(W_2) \end{array}$$

Thus, we can construct a right inverse

$$\Pi: \mathbb{Z}[K(W_1)] \otimes_{\mathbb{Z}[H^1(Y_2; \mathbb{Z})]} \otimes_{\mathbb{Z}[K(W_2)]} \cong \mathbb{Z} \left[\frac{K(W_1) \oplus K(W_2)}{H^1(Y_2; \mathbb{Z})} \right] \longrightarrow \mathbb{Z}[K(W)],$$

by defining

$$\Pi(e^w) = \begin{cases} e^v & \text{if } i(v) = w, \\ 0 & \text{otherwise.} \end{cases}$$

Tensoring the projection with M on the left, we obtain our required projection map

$$\Pi: M(W_1)(W_2) \longrightarrow M(W).$$

Theorem 3.9. Fix a module M for $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$, and let W_1 be a cobordism from Y_1 to Y_2 and W_2 be a cobordism from Y_2 to Y_3 . Let W be the composite cobordism $W = W_1 \cup_{Y_2} W_2$. Fix a Spin^c structure \mathfrak{s} over W and let $\mathfrak{s}_i = \mathfrak{s}|_{W_i}$ for $i = 1, 2$; then there are choices of representatives $\underline{F}_{W_1, \mathfrak{s}_1}^\circ \in [\underline{F}_{W_1, \mathfrak{s}_1}^\circ]$ and $\underline{F}_{W_2, \mathfrak{s}_2}^\circ \in [\underline{F}_{W_2, \mathfrak{s}_2}^\circ]$, for which

$$[\underline{F}_{W, \mathfrak{s}}^\circ] = [\Pi \circ \underline{F}_{W_2, \mathfrak{s}_2}^\circ \circ \underline{F}_{W_1, \mathfrak{s}_1}^\circ].$$

More generally, if $h \in H^1(Y_2; \mathbb{Z})$, then for the above choices of $\underline{F}_{W_1, \mathfrak{s}_1}^\circ$ and $\underline{F}_{W_2, \mathfrak{s}_2}^\circ$, we have that

$$[\underline{F}_{W, \mathfrak{s} + \delta h}^\circ] = [\Pi_M \circ \underline{F}_{W_2, \mathfrak{s}_2}^\circ \circ e^h \cdot \underline{F}_{W_1, \mathfrak{s}_1}^\circ],$$

where $\delta h \in H^2(W; \mathbb{Z})$ is the image of h under the coboundary homomorphism for the Mayer–Vietoris sequence for the decomposition of W into W_1 and W_2 .

4. Invariants of handles

We build the invariant of a cobordism up from its handle decomposition. We describe first the most interesting part—the part belonging to the two-handles (Section 4.1). After that, in Section 4.3, we consider the one- and three-handles. With these definitions in place, we define the invariant of a cobordism, and verify its topological invariance (Theorem 3.1 in the untwisted case, and Theorem 3.8) in Section 4.4. Finally we give a proof of the composition law (Theorems 3.4 and 3.9), which follows readily from the definition.

4.1. Invariants of framed links

A framed link in a three-manifold Y is a collection of n disjoint, embedded circles $K_1, \dots, K_n \subset Y$, together with a choice of homology classes $\ell_i \in H_1(\partial \text{nd}(K_i))$, with $m_i \cdot \ell_i = 1$ (where m_i is the meridian of K_i). We will often abbreviate the framed link $(K_i, \ell_i)_{i=1}^n$ by \mathbb{L} .

By attaching two-handles along the framed link \mathbb{L} , we naturally obtain a cobordism $W(\mathbb{L})$ from Y to the three-manifold $Y(\mathbb{L})$ (which is obtained by surgery along the link).

Our aim of this section is to define and study an induced map

$$\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ: \underline{HF}^\circ(Y, \mathfrak{s}|_Y, M) \longrightarrow \underline{HF}^\circ(Y(\mathbb{L}), \mathfrak{s}|_{Y(\mathbb{L})}, M(W(\mathbb{L})))$$

associated to the framed link, where \mathfrak{s} is a Spin^c structure on the cobordism $W(\mathbb{L})$.

Definition 4.1. A *bouquet for the link* \mathbb{L} is a one-complex embedded in Y which is the union of the link $\mathbb{L} = \bigcup_{i=1}^n K_i$ with a collection of paths connecting K_i to a fixed reference point in Y .

The regular neighborhood of a bouquet $B(\mathbb{L})$ is a genus n handlebody V . There is a subset of its boundary which is identified with a disjoint union of n punctured tori F_i .

Definition 4.2. A *Heegaard triple subordinate to the bouquet* $B(\mathbb{L})$ is a Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$ with the following properties:

- (1) $(\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_{n+1}, \dots, \beta_g\})$ describes the complement of the bouquet $B(\mathbb{L})$; in particular, ∂V is identified with the surface obtained from Σ by surgering out $\beta_{n+1}, \dots, \beta_g$,
- (2) $\gamma_{n+1}, \dots, \gamma_g$ are small isotopic translates of the $\beta_{n+1}, \dots, \beta_g$,
- (3) after surgering out the $\{\beta_{n+1}, \dots, \beta_g\}$, the induced curves β_i and γ_i (for $i = 1, \dots, n$) lie in the punctured torus $F_i \subset \partial V$,
- (4) for $i = 1, \dots, n$, the curves β_i represent meridians for the K_i which are disjoint from all the γ_j for $i \neq j$, and meet γ_i in a single transverse intersection point,
- (5) for $i = 1, \dots, n$ the homology classes of the γ_i correspond to the framings ℓ_i under the natural identification

$$H_1\left(\partial \text{nd}\left(\bigcup K_i\right)\right) \cong H_1(\partial V).$$

Let $X_{\alpha, \beta, \gamma}$ be the cobordism specified by the Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$.

Proposition 4.3. The manifold $X_{\alpha, \beta, \gamma}$ has three boundary components, $-Y$, $Y(\mathbb{L})$, and $\#^{g-n}(S^1 \times S^2)$. Furthermore, after filling in the third boundary by the boundary connected sum $\#^{g-n}(S^1 \times B^3)$, we obtain the standard cobordism $W(Y, \mathbb{L})$.

Proof. As a warm-up, consider the trivial case where $n = 0$ —i.e. the γ_i are all small translates of the β_i . In this case, it is easy to see that $X_{\alpha, \beta, \gamma}$ is diffeomorphic to $Y_{\alpha, \beta} \times [-1, 1]$, with a regular neighborhood of the $U_\beta \times \{0\}$ deleted. The boundary is, of course, $\#^g(S^1 \times S^2)$, so when we fill that back in, we obtain $Y \times I$.

Next, consider the case where $n = 1$. In this case,

$$Y_{\beta, \gamma} = S^3 \# \left(\#^{g-1}(S^2 \times S^1) \right),$$

where the S^3 factor is obtained by filling in β_1 and γ_1 . Now,

$$W(Y, \mathbb{L}) - \left([0, 1] \times (D^2 \times \gamma_1) \cup (D^2 \times D^2) \right) \cong Y \times [-1, 1].$$

Observe that $[0, 1] \times (D^2 \times \gamma_1) \cup (D^2 \times D^2)$ is a four-cell, attached to $Y \times I$ along a neighborhood K with the specified framing. The case for arbitrary n follows in the same manner. \square

Fix a Spin^c structure \mathfrak{s} over $W = W(Y, \mathbb{L})$. We define the map

$$\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ: \underline{HF}^\circ(Y, \mathfrak{s}|_Y, M) \longrightarrow \underline{HF}^\circ(Y(\mathbb{L}), \mathfrak{s}|_{Y(\mathbb{L})}, M(W)),$$

by

$$\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ(\zeta) = \underline{F}^\circ(\zeta \otimes \Theta, \mathfrak{s}),$$

where Θ is a generator for the top-dimensional homology $HF^{\leq 0}(\#^{g-n}(S^1 \times S^2), \mathfrak{s}_0)$. Since there are two possible generators for this latter group, the map $\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ$ is defined only up to sign. Moreover, the maps themselves are defined only up to translations by $H^1(Y; \mathbb{Z})$ and $H^1(Y(\mathbb{L}), \mathbb{Z})$. But more importantly, the map appears to depend on the bouquet and the admissible triple. However, we have the following:

Theorem 4.4. *The map $\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ$ depends only on the three-manifold Y and the framed link \mathbb{L} and the Spin^c structure \mathfrak{s} over $W(\mathbb{L})$ in the following sense. Let $(\Sigma_1, \alpha_1, \beta_1, \gamma_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, \gamma_2, z_2)$ be a pair of Heegaard triples which are subordinate to two bouquets for $\mathbb{L} \subset Y$. Then, we have a commutative diagram:*

$$\begin{array}{ccc} \underline{HF}^\circ(\alpha_1, \beta_1, \mathfrak{s}|_Y, M) & \xrightarrow{\underline{F}_{W, \mathfrak{s}}^{\circ 1}} & \underline{HF}^\circ(\alpha_1, \gamma_1, \mathfrak{s}|_{Y(\mathbb{L})}, M(W)) \\ \Psi_1^\circ \downarrow & & \Psi_2^\circ \downarrow \\ \underline{HF}^\circ(\alpha_2, \beta_2, \mathfrak{s}|_Y, M) & \xrightarrow{\underline{F}_{W, \mathfrak{s}}^{\circ 2}} & \underline{HF}^\circ(\alpha_2, \gamma_2, \mathfrak{s}|_{Y(\mathbb{L})}, M(W)) \end{array}$$

where Ψ_1° and Ψ_2° are isomorphisms induced by equivalences between the Heegaard diagrams (cf. Theorem 2.16).

The proof of this theorem occupies the rest of this section. We first show that the invariant $\underline{F}_{\mathbb{L}, \mathfrak{s}}^\circ$ is unchanged under certain operations on the Heegaard diagram (which leave the bouquet unchanged). These operations are given in the following:

Lemma 4.5. *Let Y be a closed, oriented three-manifold equipped with a framed link $\mathbb{L} \subset Y$ and associated bouquet B . Then, there is a Heegaard triple subordinate to B , and indeed any two subordinate pointed Heegaard triples can be connected by a sequence of the following moves:*

- (1) handleslides and isotopies amongst the $\{\alpha_1, \dots, \alpha_g\}$,
- (2) handleslides and isotopies amongst the β_i for $i = n + 1, \dots, g$, carrying along the γ_i for $i = n + 1, \dots, g$, as well,

- (3) isotopies and possible handleslides of some β_i for $i \in 1, \dots, n$ across β_j for $j \in n+1, \dots, g$,
- (4) isotopies and possible handleslides of the γ_i for $i \in 1, \dots, n$ across the γ_j for $j \in n+1, \dots, g$,
- (5) stabilizations (introducing α_{g+1} , β_{g+1} , and γ_{g+1}).

As usual, isotopies and handleslides here take place in the complement of the basepoint z .

Proof. Fix a Morse function on $B(\mathbb{L})$ with one critical point of index three and n index two critical points. Let $\{\beta_1, \dots, \beta_n\}$ be the attaching circles for these index two critical points.

We complete the above to a triple extending this Morse function over all of $Y - B(\mathbb{L})$ (introducing only one index zero, and no new index three critical points). This gives rise to Σ , $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_{n+1}, \dots, \beta_g\}$ in the usual manner. Let z then be any basepoint in the complement of these curves. Curves $\{\gamma_{n+1}, \dots, \gamma_g\}$ can be found then by taking small translates of the corresponding β_i .

Now, if we have two such Morse functions (giving rise to Heegaard triples subordinate to a fixed bouquet), we can connect the Morse functions through a generic one-parameter family. This allows us to connect the two handle decompositions for $Y - B(\mathbb{L})$ through a sequence of handle-slides, pair creations and annihilations.

In this process, we possibly introduce new index zero and index three critical points which cancel new attaching circles α_i and β_j . It is straightforward to see that no matter how these cancellations occur, the two handle decompositions differ by only isotopies and handleslides amongst the $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_{n+1}, \dots, \beta_g\}$, and also stabilizations (cf. [16], Proposition 2.2).

Observe that if we surger $\{\beta_{n+1}, \dots, \beta_g\}$ out of Σ , the remaining two-manifold is identified with $\partial B(\mathbb{L})$; in particular, we have candidates for the curves $\{\gamma_1, \dots, \gamma_n\}$ in $\partial B(\mathbb{L})$, which we can perturb slightly so that they induce curves in Σ .

Next, we consider the dependence of the construction on the β_i and γ_i for $i = 1, \dots, n$. The link specifies the homology classes of the β_i in F_i , while the framing specifies the homology classes of the γ_i . Different choices β_i and β'_i can be connected by an isotopy in F_i . It follows that in Σ they can be connected by a sequence of isotopies and handleslides across the $\{\beta_{n+1}, \dots, \beta_g\}$. The γ_i for $i = 1, \dots, n$ follow in the same manner.

Since $\Sigma - \beta_1 - \dots - \beta_g - \gamma_1 - \dots - \gamma_n$ is connected, we can realize moving the basepoint as a sequence of handleslides amongst the α_i (cf. Section 9 of [16]). \square

We have the following special case of Theorem 4.4:

Proposition 4.6. *Fix a framed link $\mathbb{L} \subset Y$, a Spin^c structure $\mathfrak{s} \in W(\mathbb{L})$, and a bouquet B for \mathbb{L} . Then, the map $F_{\mathbb{L}, \mathfrak{s}}^\circ$ is independent of the underlying Heegaard triple subordinate to the bouquet, in the sense that if $(\Sigma_1, \alpha_1, \beta_1, \gamma_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, \gamma_2, z_2)$ are a pair of Heegaard triples which are subordinate to a fixed bouquet $B(\mathbb{L})$ for $\mathbb{L} \subset Y$.*

Then, we have a commutative diagram:

$$\begin{array}{ccc} \underline{HF}^\circ(\alpha_1, \beta_1, \mathfrak{s}|Y, M) & \xrightarrow{F_{W, \mathfrak{s}}^{\circ, 1}} & \underline{HF}^\circ(\alpha_1, \gamma_1, \mathfrak{s}|Y(\mathbb{L}), M(W)) \\ \Psi_1^\circ \downarrow & & \Psi_2^\circ \downarrow \\ \underline{HF}^\circ(\alpha_2, \beta_2, \mathfrak{s}|Y, M) & \xrightarrow{F_{W, \mathfrak{s}}^{\circ, 2}} & \underline{HF}^\circ(\alpha_2, \gamma_2, \mathfrak{s}|Y(\mathbb{L}), M(W)) \end{array}$$

where Ψ_1° and Ψ_2° are isomorphisms induced by equivalences between the Heegaard diagrams.

This proposition is divided into steps, the first of which is a stabilization invariance for holomorphic triples, which in turn follows from the gluing result for holomorphic triangles, Theorem 2.14.

Lemma 4.7. *The map $F_{\mathbb{L}, \mathfrak{s}}^\circ$ commutes with the isomorphisms induced by stabilizations (introducing α_{g+1} , β_{g+1} , and γ_{g+1}).*

Proof. Start with a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$. The stabilization involves forming the connected sum of Σ with a genus one surface E containing: a pair of curves α_{g+1} and β_{g+1} (which meet in a single, transverse intersection point), and a new curve γ_{g+1} which is a small isotopic copy of β_{g+1} meeting it in two transverse intersection points (and meeting α_g in a single transverse intersection point). Then, the new Heegaard triple

$$(\Sigma, \{\alpha_1, \dots, \alpha_{g+1}\}, \{\beta_1, \dots, \beta_{g+1}\}, \{\gamma_1, \dots, \gamma_{g+1}\}, z)$$

represents a stabilization of the original Heegaard triple.

Let $x_0 = \alpha_{g+1} \cap \beta_{g+1}$, $w_0 = \alpha_{g+1} \cap \gamma_{g+1}$, and let y_0 be the intersection point of β_{g+1} with γ_{g+1} with higher relative degree. It is easy to see that there is a unique homotopy class of triangle $\psi_0 \in \pi_2(x_0, y_0, w_0)$ with nowhere negative coefficients, and moreover $\mu(\psi_0) = 0$, and ψ_0 has a unique, smooth holomorphic representative (if we take a constant complex structure on E), see Fig. 1. Clearly, if the $\Theta \in HF^{\leq 0}(\mathbb{T}_\beta, \mathbb{T}_\gamma, \mathfrak{s}_0)$ represents a top-dimensional generator, where $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g$ and $\mathbb{T}_\gamma = \gamma_1 \times \dots \times \gamma_g$, then $\Theta' = \Theta \times \{y_0\} \in HF^+(\mathbb{T}_\beta \times \beta_{g+1}, \mathbb{T}_\gamma \times \gamma_{g+1}, \mathfrak{s}_0)$.

Consider the square

$$\begin{array}{ccc} CF^+(\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathfrak{s}|Y) & \xrightarrow{F_1^+} & CF^+(\mathbb{T}_\alpha, \mathbb{T}_\gamma, \mathfrak{s}|Y(\mathbb{L})) \\ \cong \downarrow & & \downarrow \cong \\ CF^+(\mathbb{T}_\alpha \times \alpha_{g+1}, \mathbb{T}_\beta \times \beta_{g+1}) & \xrightarrow{F_2^+} & CF^+(\mathbb{T}_\alpha \times \alpha_{g+1}, \mathbb{T}_\beta \times \gamma_{g+1}) \end{array}$$

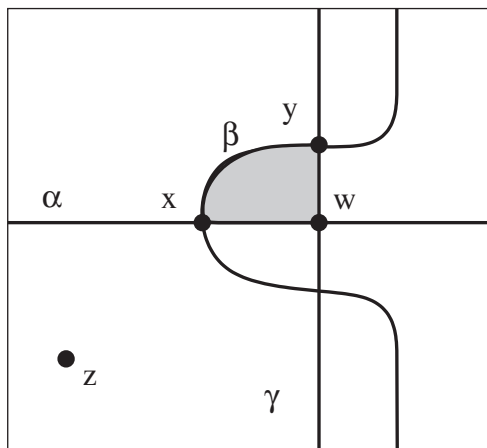


Fig. 1. Stabilization: We have pictured here the torus E used in stabilizing. (We have dropped all the subscripts from the picture.) The connected sum point is labelled z . The domain of ψ_0 is lightly shaded in the picture.

where the vertical isomorphisms are induced by the maps $\mathbf{x} \in \text{Sym}^g(\Sigma - \text{nd}(z)) \mapsto \mathbf{x} \times x_0 \in \text{Sym}^{g+1}(\Sigma \# E)$ and $\mathbf{y} \mapsto \mathbf{y} \times y_0$. For all sufficiently long connected sum neck, these induce isomorphisms of chain complexes, according to Theorem 10.2 of [16]. Theorem 2.14 gives a map

$$\pi_2(\mathbf{x}, \Theta, \mathbf{w}) \longrightarrow \pi_2(\mathbf{x} \times x_0, \Theta', \mathbf{w} \times w_0)$$

and an identification of formally zero-dimensional moduli spaces $\mathcal{M}(\psi) \cong \mathcal{M}(\psi')$, which shows that the square exhibited above commutes. \square

Proof of Proposition 4.6. First, we organize Lemma 4.5 in the same manner as Lemma 2.10: if $(\Sigma_1, \alpha_1, \beta_1, \gamma_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, \gamma_2, z_2)$ are a pair of Heegaard triples, both of which are subordinate to a fixed bouquet $B(\mathbb{L})$ for a framed link $\mathbb{L} \subset \mathbb{K}$, then we can go from one to the other by first a sequence of stabilizations, then a strong equivalence—i.e. using only Moves (1)–(4) of Lemma 4.5 (in particular leaving the underlying two-manifold Σ unchanged)—and then a sequence of destabilizations.

Having shown (in Lemma 4.7) that $\underline{F}_{\mathbb{L},s}^\circ$ commutes with the isomorphism induced by stabilization, it suffices to show that $\underline{F}_{\mathbb{L},s}^\circ$ commutes with the isomorphism induced by a strong equivalence.

Suppose that $(\Sigma, \alpha_1, \beta_1, \gamma_1, z)$ and $(\Sigma, \alpha_2, \beta_2, \gamma_2, z)$ are strongly equivalent. Then it is easy to see that α_2 , β_2 , and γ_2 are obtained by handleslides and isotopies amongst the α_1 , β_1 , and γ_1 , respectively. Following the scheme from Section 2.5, we choose isotopic copies α'_1 , β'_1 and γ'_1 of the α_1 , β_1 , and γ_1 , so that each of $(\Sigma, \alpha_1, \alpha'_1, z)$, $(\Sigma, \beta_1, \beta'_1, z)$ and $(\Sigma, \gamma, \gamma'_1, z)$ is admissible. Now the fact that $F_{\mathbb{K},s}$ commutes with the isomorphism of the strong equivalence follows from the commutativity of the following two squares.

First, we have

$$\begin{array}{ccc} HF^\circ(\alpha_1, \beta_1) & \xrightarrow{\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}} & HF^\circ(\alpha'_1, \beta'_1) \\ F^\circ(\cdot \otimes \Theta_{\beta_1, \gamma_1}, \mathfrak{s}) \downarrow & & \downarrow F^\circ(\cdot \otimes \Theta_{\beta'_1, \gamma'_1}, \mathfrak{s}) \\ HF^\circ(\alpha_1, \gamma_1) & \xrightarrow{\Gamma_{\alpha'_1, \alpha_1; \gamma_1, \gamma'_1}} & HF^\circ(\alpha'_1, \gamma'_1) \end{array}$$

This square commutes up to sign since isotopy invariance of the triangle construction gives that

$$\begin{aligned} \Gamma_{\alpha'_1, \alpha_1; \gamma_1, \gamma'_1} \circ F_{\alpha_1, \beta_1, \gamma_1}^\circ(\zeta \otimes \Theta_{\beta_1, \gamma_1}, \mathfrak{s}) &= F_{\alpha'_1, \beta_1, \gamma'_1}^\circ(\Gamma_{\alpha'_1, \alpha_1; \beta_1}(\zeta) \otimes \Gamma_{\beta_1, \gamma_1, \gamma'_1}(\Theta_{\beta_1, \gamma_1}), \mathfrak{s}) \\ &= \pm F_{\alpha'_1, \beta_1, \gamma'_1}^\circ(\Gamma_{\alpha'_1, \alpha_1; \beta_1}(\zeta) \otimes \Theta_{\beta_1, \gamma'_1}) \\ &= \pm F_{\alpha'_1, \beta_1, \gamma'_1}^\circ(\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}(\zeta) \otimes \Gamma_{\beta'_1, \beta_1, \gamma'_1}(\Theta_{\beta_1, \gamma'_1})) \\ &= \pm F_{\alpha'_1, \beta_1, \gamma'_1}^\circ(\Gamma_{\alpha'_1, \alpha_1; \beta_1, \beta'_1}(\zeta) \otimes \Theta_{\beta'_1, \gamma'_1}), \end{aligned}$$

in view of the fact that $\Gamma_{\beta_1, \gamma_1, \gamma'_1}(\Theta_{\beta_1, \gamma'_1}) = \pm \Theta_{\beta_1, \gamma'_1}$, and $\Gamma_{\beta'_1, \beta_1, \gamma'_1}(\Theta_{\beta_1, \gamma'_1}) = \pm \Theta_{\beta'_1, \gamma'_1}$ (since both sides of both equations are generators for a group—the top-dimensional homology of $HF^{\leq 0}(\#^{g-1}(S^1 \times S^2), \mathfrak{s}_0)$ —which is isomorphic to \mathbb{Z}).

The second commutative square is

$$\begin{array}{ccc} HF^\circ(\alpha'_1, \beta'_1) & \xrightarrow{\Theta_{\alpha_2, \alpha'_1} \otimes \cdot \otimes \Theta_{\beta'_1, \beta_2}} & HF^\circ(\alpha_2, \beta_2) \\ \otimes \Theta_{\beta'_1, \gamma'_1} \downarrow & & \downarrow \otimes \Theta_{\beta_2, \gamma_2} \\ HF^\circ(\alpha'_1, \gamma'_1) & \xrightarrow{\Theta_{\alpha_2, \alpha'_1} \otimes \cdot \otimes \Theta_{\gamma'_1, \gamma_2}} & HF^\circ(\alpha_2, \gamma_2) \end{array}$$

which now commutes up to sign, owing to the associativity of the triangle construction (Theorem 2.5), and now the fact that

$$F^{\leq 0}(\Theta_{\beta'_1, \gamma'_1} \otimes \Theta_{\gamma'_1, \gamma_2}, \mathfrak{s}_0) = \pm \Theta_{\beta'_1, \gamma_2} = \pm F^{\leq 0}(\Theta_{\beta'_1, \beta_2} \otimes \Theta_{\beta_2, \gamma_2}, \mathfrak{s}_0). \quad \square$$

Lemma 4.8. *The map $F_{\mathbb{L}, \mathfrak{s}}^\circ$ is independent of the bouquet.*

Proof. Suppose that B and B' are a pair of bouquets which differ in the choice of one path σ_1 and σ'_1 . We construct two Heegaard triples

$$(\Sigma, \alpha, \beta, \gamma, z) \quad \text{and} \quad (\Sigma, \alpha, \beta', \gamma', z)$$

in such a manner that the sets β' are obtained from the β by a sequence of handleslides, and the γ' are obtained from the γ by a sequence of handleslides.

To this end, consider the regular neighborhood M of $B \cup B'$. This is a handlebody of genus $n + 1$. On the complement $Y - M$, we have a Morse function with one index zero critical point, g index one, and $g - n - 1$ index two critical points. Let $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_{n+2}, \dots, \beta_g\}$ be the corresponding attaching circles. Let $\{\beta_1, \dots, \beta_n\}$ be attaching circles in ∂M for the index two critical points corresponding to reattaching the components K_1, \dots, K_n of the link. There is one additional index two critical point on M . Let β_{n+1} be the attaching circle dual to σ_1 : this is the attaching circle for a Morse function on $M - \text{nd}(B)$ with a single index two critical point. Similarly, let β'_{n+1} be the attaching circle dual to σ' (while all other $\beta'_i = \beta_i$). We can arrange for β'_{n+1} to be disjoint from β_{n+1} . Let γ_i for $i = 1, \dots, n$ correspond to the given framing of the link, and γ_i be small isotopic translates of the β_i for $i = n + 1, \dots, g$, while γ'_i is a small isotopic translate of β'_i (while all other $\gamma'_i = \gamma_i$), then, $(\Sigma, \alpha, \beta, \gamma, z)$ is subordinate to $B(\mathbb{L})$, while $(\Sigma, \alpha, \beta', \gamma', z)$ is subordinate to $B(\mathbb{L}')$.

Now, if we surger $\{\beta_1, \dots, \beta_{n-1}, \beta_{n+1}, \dots, \beta_g\}$ out of Σ , we obtain a surface of genus one, with two disjoint, embedded, homologically non-trivial curves induced by β_n and β'_n . These curves must then be isotopic in the torus: thus, β'_n can be obtained by handlesliding β_n over some collection of the $\{\beta_1, \dots, \beta_{n-1}, \beta_{n+1}, \dots, \beta_g\}$. The analogous remark applies to obtaining γ'_n from γ_n . Thus, β' are obtained from β by handleslides and isotopies, and γ' are obtained from γ by handleslides and isotopies.

The result then follows from the usual commutative diagram

$$\begin{array}{ccc} \underline{HF}^\circ(\alpha, \beta, \mathfrak{s}|_Y, M) & \xrightarrow{F^\circ(\cdot \otimes \Theta_{\beta, \gamma}, \mathfrak{s})} & \underline{HF}^\circ(\alpha, \gamma, \mathfrak{s}|_{Y(\mathbb{L})}, M(W)) \\ \underline{F}^\circ(\cdot \otimes \Theta_{\beta, \beta'}, \mathfrak{s}) \downarrow & & \downarrow \underline{F}^\circ(\cdot \otimes \Theta_{\gamma, \gamma'}, \mathfrak{s}) \\ \underline{HF}^\circ(\alpha, \beta', \mathfrak{s}|_Y, M) & \xrightarrow{F^\circ(\cdot \otimes \Theta_{\beta', \gamma'}, \mathfrak{s})} & \underline{HF}^\circ(\beta', \gamma', \mathfrak{s}|_{Y(\mathbb{L})}, M(W)) \end{array}$$

(where the vertical maps are isomorphisms induced by strong equivalence, and the two horizontal maps are the two candidates for $\underline{F}^\circ_{\mathbb{L}, \mathfrak{s}}$ belonging to the two bouquets), and use of associativity, according to which we have the following equation up to sign

$$F^{\leq 0}(\Theta_{\beta, \gamma} \otimes \Theta_{\gamma, \gamma'}, \mathfrak{s}_0) = \pm \Theta_{\beta, \gamma'} = \pm F^{\leq 0}(\Theta_{\beta, \beta'} \otimes \Theta_{\beta', \gamma'}, \mathfrak{s}_0). \quad \square$$

Proof of Theorem 4.4. This is now an immediate consequence of Proposition 4.6, together with Lemma 4.8. \square

4.2. Compositions of link invariants

The invariant $\underline{F}^\circ_{\mathbb{L}, \mathfrak{s}}$ satisfies the following analogue of Theorem 3.4.

If we partition the link $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$, this gives a decomposition of the cobordism $W(\mathbb{L})$ as a union

$$W(\mathbb{L}) = W_1 \cup_{Y(\mathbb{L}_1)} W_2,$$

where $W_1 = W(Y, \mathbb{L}_1)$, and W_2 is the cobordism from $Y(\mathbb{L}_1)$ to $Y(\mathbb{L})$ associated to \mathbb{L}_2 , thought of as a framed link in $Y(\mathbb{L}_1)$. Recall that we have a projection map

$$\Pi: \underline{HF}^\circ(Y(\mathbb{L}), M(W_1)(W_2)) \longrightarrow \underline{HF}^\circ(Y(\mathbb{L}), M(W)).$$

Proposition 4.9. *There are choices $\underline{F}_{W_1, \mathbb{S}_1}^\circ$ and $\underline{F}_{W_2, \mathbb{S}_2}^\circ$ from the equivalence class of maps $\underline{F}_{Y, \mathbb{L}_1, \mathbb{S}|W_1}^\circ$ and $\underline{F}_{Y(\mathbb{L}_1), \mathbb{L}_2}^\circ$ respectively, with the property that the projection of the composite map $\Pi \circ \underline{F}_{W_2}^\circ \circ \underline{F}_{W_1}^\circ$ represents $\underline{F}_{Y, \mathbb{L}}^\circ$.*

Proof. Let \mathbb{L}_1 and \mathbb{L}_2 have m and n components, respectively. Let $\{\beta_1, \dots, \beta_{n+m}\}$ correspond to the link $\mathbb{L}_1 \cup \mathbb{L}_2$. Let δ_i be small isotopic translates of the γ_i for $i = 1, \dots, m$ correspond to the framings of \mathbb{L}_1 , and let δ_i be small isotopic translates of β_i for $i = m + 1, \dots, m + n$. Clearly,

$$\underline{F}_{Y, \mathbb{L}_1}^\circ = \underline{F}^\circ(\cdot \otimes \Theta_{\beta, \delta, \mathbb{S}}, M): \underline{HF}^\circ(\alpha, \beta, \mathbb{S}|_Y, M) \longrightarrow \underline{HF}^\circ(\alpha, \delta, \mathbb{S}|_{Y(\mathbb{L}_1)}, M(W_1)),$$

while

$$\underline{F}_{Y(\mathbb{L}_1); \mathbb{L}_2}^\circ = \underline{F}^\circ(\cdot \otimes \Theta_{\delta, \gamma, \mathbb{S}}, M(W_1)): \underline{HF}^\circ(\alpha, \delta, \mathbb{S}|_{Y(\mathbb{L}_1)}, M(W_1)(W_2)) \longrightarrow \underline{HF}^\circ(\alpha, \gamma, \mathbb{S}|_{Y(\mathbb{L}_1 \cup \mathbb{L}_2)}),$$

Next, we claim that

$$F^{\leq 0}(\Theta_{\beta, \delta} \otimes \Theta_{\delta, \gamma, \mathbb{S}_0}) = \pm \Theta_{\beta, \gamma}.$$

This follows from Theorem 2.14, and two model calculations in the genus one surface E . In one case, we have three curves β , γ , and δ which are all three Hamiltonian isotopic translates of one another, as in Fig. 2 below (with different labellings); in the other case, β and δ are small isotopic translates of one another, and γ intersects β (and also δ) in a single, transverse intersection point (we met this configuration already in the proof of stabilization invariance, see Fig. 1, only with different notation). In each case, there is a unique homotopy class with nowhere negative coefficients connecting the relevant intersection points, and the homotopy class supports a single smooth solution.

The theorem now follows from associativity in the twisted case. Note that in the present application of associativity, we consider the pointed Heegaard quadruple $(\Sigma, \alpha, \beta, \delta, \gamma, z)$, and as such we must verify that

$$\delta H^1(Y_{\alpha, \delta})|_{Y_{\beta, \gamma}} = 0 = \delta H^1(Y_{\beta, \gamma})|_{Y_{\alpha, \delta}}.$$

Both follow from the fact that the map on $H_2(Y_{\beta, \gamma}) \longrightarrow H_2(X_{\alpha, \beta, \delta, \gamma})$ is trivial (to see this, observe that the (β, γ) -periodic domains, which evidently involve relations amongst the β_i and γ_j for $i, j \in \{m + 1, \dots, m + n\}$, can be expressed as sums of (β, γ) - and (β, δ) -periodic domains). \square

4.3. One- and three-handles

Let U be the cobordism obtained by adding a single one-handle to a connected three-manifold Y . This is a cobordism between Y and $Y' = Y \# (S^1 \times S^2)$. Clearly, $\text{Spin}^c(U) \cong \text{Spin}^c(Y)$; moreover, the Spin^c structures over Y' which extends over U are those which have trivial first Chern class on the $S^1 \times S^2$ factor.

We define the invariant for the one-handle addition as follows. Fix a standard Heegaard diagram (E, α, β, z_0) for $S^2 \times S^1$ (in the sense of Definition 2.8) so that α and β meet in a pair of intersection points; and let θ be the one with higher relative grading. Given a Heegaard decomposition $(\Sigma, \alpha, \beta, z)$ for Y , let $(\Sigma', \alpha', \beta', z') = (\Sigma, \alpha, \beta, z) \# (E, \alpha, \beta, z_0)$ be an associated Heegaard decomposition for $Y \# (S^2 \times S^1)$. Then, we define

$$\underline{g}_{U, \mathfrak{s}}^\infty: \underline{CF}^\infty(\alpha, \beta, \mathfrak{s}, M) \longrightarrow \underline{CF}^\infty(\alpha', \beta', \mathfrak{s} \# \mathfrak{s}_0, M)$$

be the map induced from

$$\underline{g}_{U, \mathfrak{s}}^\infty(m \otimes [\mathbf{x}, i]) = m \otimes [\mathbf{x} \times \{\theta\}, i].$$

(Observe that $M(U) \cong M$.) When the complex structure on $\Sigma \# E$ is sufficiently stretched out, this is a chain map (cf. [15], Proposition 6.4), and we define the map associated to the one-handle

$$\underline{G}_{U, \mathfrak{s}}^\circ: \underline{HF}^\circ(Y, \mathfrak{s}|Y, M) \longrightarrow \underline{HF}^\circ(Y', \mathfrak{s}|Y', M)$$

be the induced map on homology.

Theorem 4.10. *The maps $\underline{G}_{U, \mathfrak{s}}^\circ$ depends only on the three-manifold Y and the Spin^c structure \mathfrak{s} in the following sense. If $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ are equivalent Heegaard diagrams, then the corresponding diagrams $(\Sigma'_1, \alpha'_1, \beta'_1, z'_1)$ and $(\Sigma'_2, \alpha'_2, \beta'_2, z'_2)$ are equivalent, and indeed we have a commutative diagram*

$$\begin{array}{ccc} HF^\circ(\alpha_1, \beta_1, \mathfrak{t}) & \xrightarrow{\underline{G}_{U, \mathfrak{s}}^\circ} & HF^\circ(\alpha'_1, \beta'_1, \mathfrak{t} \# \mathfrak{s}_0) \\ \Psi_Y^\circ \downarrow & & \downarrow \Psi_{Y \# (S^2 \times S^1)}^\circ \\ HF^\circ(\alpha_2, \beta_2, \mathfrak{t}) & \xrightarrow{\underline{G}_{U, \mathfrak{s}}^\circ} & HF^\circ(\alpha'_2, \beta'_2, \mathfrak{t} \# \mathfrak{s}_0) \end{array}$$

where the vertical maps are the isomorphisms induced from the equivalences of the Heegaard diagrams (cf. Theorem 2.1).

Proof. The equivalence of $(\Sigma, \alpha_1, \beta_1, z_1)$ and $(\Sigma, \alpha_2, \beta_2, z_2)$ induces the equivalence between the corresponding diagrams $(\Sigma_1, \alpha'_1, \beta'_1, z'_1)$ and $(\Sigma_2, \alpha'_2, \beta'_2, z'_2)$ in view of the

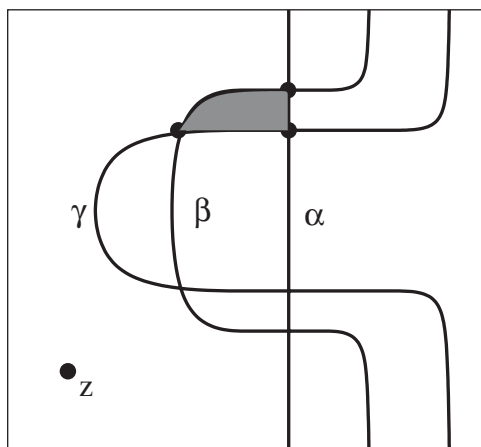


Fig. 2. One-handles: Three curves in the torus, obtained as small isotopic translates of one another. The domain of holomorphic triangle connecting the top-dimensional intersection points is lightly shaded.

fact that an isotopy of an attaching circle in a Heegaard diagram for Y corresponds to a pair of handleslides across α_{g+1} or β_{g+1} supported in E for the induced Heegaard diagram of $Y\#(S^2 \times S^1)$.

The fact that $G_{U,5}^\infty$ commutes with the isomorphism induced by equivalence, follows from a gluing analogous to Lemma 4.7, only now the corresponding picture in the torus E is different; see Fig. 2. Here, α , β , and γ are all isotopic translates of the same curve (the curve α_{g+1} in E). \square

Dually, if V is the cobordism obtained by adding a single three-handle along a non-separating two-sphere $S \subset Y'$, then Y' can be written as $Y\#(S^2 \times S^1)$ and V is a cobordism from $Y' = Y\#(S^2 \times S^1)$ to Y . There is a special kind of compatible Heegaard diagram induced by the embedded sphere.

Lemma 4.11. *Let $S \subset Y'$ be a non-separating embedded sphere in a three-manifold, then there is an induced split Heegaard diagram $(\Sigma', \alpha', \beta', z')$ for Y' of the form*

$$(\Sigma', \alpha', \beta', z') = (\Sigma, \alpha, \beta, z) \# (E, \alpha, \beta, z_0),$$

where (E, α, β, z_0) is a standard Heegaard diagram of $S^2 \times S^1$ (and where the sphere S is represented in this factor). Moreover, if we have two such split diagrams which are equivalent,

$$(\Sigma_1, \alpha_1, \beta_1, z_1) \# (E, \alpha, \beta, z_0) \quad \text{and} \quad (\Sigma_2, \alpha_2, \beta_2, z_2) \# (E, \alpha, \beta, z_0),$$

then $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ are equivalent Heegaard diagrams.

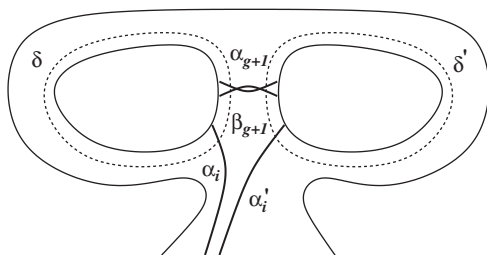


Fig. 3. Two-spheres splitting Heegaard diagrams: This is an illustration of the uniqueness claim in Lemma 4.11. The curves α_{g+1} and β_{g+1} bound a periodic domain representing the embedded two-sphere, and the curves δ and δ' give two candidates for the splitting of E . Handlesliding α_i across α_{g+1} , we get a curve α'_i which is isotopic to α_i , after we surger out α_{g+1} .

Proof. We build the Heegaard diagram starting from a handle decomposition of the neighborhood of the two-sphere, as in Lemma 7.3 of [15] (where we were interested in the case of embedded surfaces of genus $g > 0$, though the construction works the same way when $g = 0$). Specifically, we start with Morse function on the neighborhood of the two-sphere S , and consider extensions to the complement $Y - S$. In this manner, we obtain a Heegaard diagram $(\Sigma', \alpha', \beta', z')$, with a periodic domain representing S which is bounded by curves α_{g+1} and β_{g+1} which are small translates of one another. Indeed, in the present case, we arrange that β_{g+1} is a small Hamiltonian translate of α_{g+1} . Now, any two Heegaard diagrams which arise in this manner are equivalent, through an equivalence which leaves α_{g+1} and β_{g+1} unchanged.

Pick any curve δ dual to α_{g+1} (and β_{g+1}). It is easy to see that after handleslides across the α_{g+1} , we can arrange for all the remaining $\{\alpha_1, \dots, \alpha_g\}$ to be disjoint from γ . Similarly, handlesliding across β_{g+1} , we arrange for the $\{\beta_1, \dots, \beta_g\}$ to be disjoint from δ . The curves δ and α_{g+1} and β_{g+1} lie in a torus summand of Σ' , giving us the required splitting.

To verify uniqueness of the splitting, suppose we have two different such dual curves δ and δ' . In the first case, we destabilize along the neighborhoods $\alpha_{g+1} \cup \delta$ (i.e. surgering out a regular neighborhood), and in the second we destabilize along $\alpha_{g+1} \cup \delta'$. Both two-manifolds are identified with the two-manifold obtained by surgering out α_{g+1} . To see that the two induced Heegaard diagrams are equivalent, we make the following observation. Suppose that α_i is a curve which meets δ , but misses δ' . By handlesliding repeatedly across α_{g+1} , we obtain a new curve α'_i which meets δ' , but is disjoint from δ . However, if we then surger out α_{g+1} , the induced curves (images of α_i and α'_i are isotopic. (See Fig. 3.) \square

We now take a split Heegaard diagram

$$(\Sigma', \alpha', \beta', z') = (\Sigma, \alpha, \beta, z) \# (E, \alpha, \beta, z_0)$$

as in the above lemma. Then, we define

$$\underline{e}_{V, \mathfrak{s}}^\infty: CF^\infty(\alpha', \beta', \mathfrak{s} | Y') \longrightarrow CF^\infty(\alpha, \beta, \mathfrak{s} | Y)$$

to be the map

$$\begin{aligned} & \underline{e}_{V,\mathfrak{s}}^\infty[\mathbf{x} \times \{y\}, i] \\ &= \begin{cases} [\mathbf{x}, i] & \text{if } y \text{ is the minimal intersection point for the } S^2 \times S^1 \text{ factor,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Again, if the complex structure on $\Sigma \# E$ is sufficiently stretched, out, $\underline{e}_{V,\mathfrak{s}}^\infty$ gives a chain map, inducing the required map on homology

$$\underline{E}_{V,\mathfrak{s}}^\circ: \underline{HF}^\circ(Y', \mathfrak{s}|Y', M) \longrightarrow \underline{HF}^\circ(Y, \mathfrak{s}|Y, M).$$

The following analogue of Theorem 4.10 holds for three-handles.

Theorem 4.12. *Fix a non-separating embedded sphere $S \subset Y$, and let*

$$(\Sigma'_1, \alpha'_1, \beta'_1, z'_1) = (\Sigma_1, \alpha_1, \beta_1, z_1) \# (E, \alpha, \beta, z_0)$$

and

$$(\Sigma'_2, \alpha'_2, \beta'_2, z'_2) = (\Sigma_2, \alpha_2, \beta_2, z_2) \# (E, \alpha, \beta, z_0)$$

be a pair of equivalent split Heegaard diagrams, then the following square commutes:

$$\begin{array}{ccc} HF^\circ(\alpha'_1, \beta'_1, \mathfrak{t} \# \mathfrak{s}_0) & \xrightarrow{\underline{E}_{V,\mathfrak{s}}^\circ} & HF^\circ(\alpha_1, \beta_1, \mathfrak{t}) \\ \Psi_{Y'}^\circ \downarrow & & \downarrow \Psi_Y^\circ \\ HF^\circ(\alpha'_2, \beta'_2, \mathfrak{t} \# \mathfrak{s}_0) & \xrightarrow{\underline{E}_{V,\mathfrak{s}}^\circ} & HF^\circ(\alpha_2, \beta_2, \mathfrak{t}) \end{array}$$

where the vertical maps are the isomorphisms induced by the equivalences.

Proof. This follows exactly as in the case for one-handles, Theorem 4.10: i.e. we apply the gluing of Theorem 2.14 for the appropriate triangle in the torus with three curves illustrated in Fig. 2. \square

4.4. Invariance of the maps associated to cobordisms

We prove Theorems 3.8 and 3.1.

Let W be a cobordism from Y_1 to Y_2 . We decompose $W = W_1 \#_{Y'_1} W_2 \#_{Y'_2} W_3$, where W_1 is a collection of one-handles, W_2 is a collection of two-handles, and W_3 is a

collection of three-handles. Concretely, W_2 can be represented as a framed link $\mathbb{L} \subset Y'_1 = Y_1 \# (\#^\ell(S^2 \times S^1))$. We then define

$$\underline{F}_{W,\mathfrak{s}}^\circ = \underline{E}_{W_3,\mathfrak{s}}^\circ \circ \underline{F}_{\mathbb{L},\mathfrak{s}}^\circ \circ \underline{G}_{W_1,\mathfrak{s}}^\circ,$$

where $\underline{E}_{W_3,\mathfrak{s}}^\circ$ and $\underline{G}_{W_1,\mathfrak{s}}^\circ$ are defined to be composites of the maps \underline{E}° and \underline{G}° induced by the various one- and three-handles.

To see that this depends on the underlying four-manifold only (not on the particular handle decomposition), we proceed in several steps.

Lemma 4.13. *The maps $\underline{G}_{W_1,\mathfrak{s}}^\circ$ are invariant under the ordering of the one-handles, and handleslides amongst them.*

Proof. We verify that $\underline{G}_{W_1,\mathfrak{s}}^\circ$ is independent of the ordering of the one-handles. Addition of a pair of two one-handles to a Heegaard diagram corresponds to adding two-handles to the Heegaard surface to obtain Σ' , and placing a pair of new curves α_1 and β_1 and α_2 and β_2 , where α_i is a small isotopic translate of β_i , and both pairs are supported inside the two new handles. From this description, it is clear that the composite of the two one-handle additions is independent of their order. Handleslide invariance follows from this. \square

Lemma 4.14. *Fix a framed link \mathbb{L} in Y , and let \mathbb{L}' be the framed link obtained by handleslides amongst the components of \mathbb{L} . Then, the maps $\underline{F}_{Y;\mathbb{L}}^\circ$ and $\underline{F}_{Y;\mathbb{L}'}^\circ$ are equal.*

Proof. Let K'_1 be obtained from K_1 by a handleslide over K_2 . To perform this handleslide, one needs a path σ joining K_1 to K_2 . After the handleslide, there is a natural path σ' joining K'_1 and K_2 .

Complete $K_1 \cup \sigma \cup K_2$ to a bouquet for \mathbb{L} . Let $(\Sigma, \alpha, \beta, \gamma, z)$ be the associated triple, where β_1 is dual to K_1 , β_2 is dual to K_2 . We can complete $K'_1 \cup \sigma' \cup K_2$ to a bouquet for \mathbb{L}' so that there is a subordinate Heegaard triple

$$(\Sigma', \alpha, \beta', \gamma', z)$$

with the property that β'_2 is obtained as a handleslide of β_2 over β_1 , γ'_1 is obtained as a handleslide of γ_1 over γ_2 , and all other $\beta'_i = \beta_i$, $\gamma'_i = \gamma_i$ (cf. Fig. 4).

We then have the following commutative diagram:

$$\begin{array}{ccc} \underline{HF}^\circ(Y, \mathfrak{s}|Y) & \xrightarrow{\otimes \Theta_{\beta,\gamma}} & \underline{HF}^\circ(Y(\mathbb{L}), \mathfrak{s}|Y(\mathbb{L})) \\ \otimes \Theta_{\beta,\beta'} \downarrow & & \otimes \Theta_{\gamma,\gamma'} \downarrow \\ \underline{HF}^\circ(Y, \mathfrak{s}|Y) & \xrightarrow{\otimes \Theta_{\beta',\gamma'}} & \underline{HF}^\circ(Y(\mathbb{L}'), \mathfrak{s}|Y(\mathbb{L}')) \end{array}$$

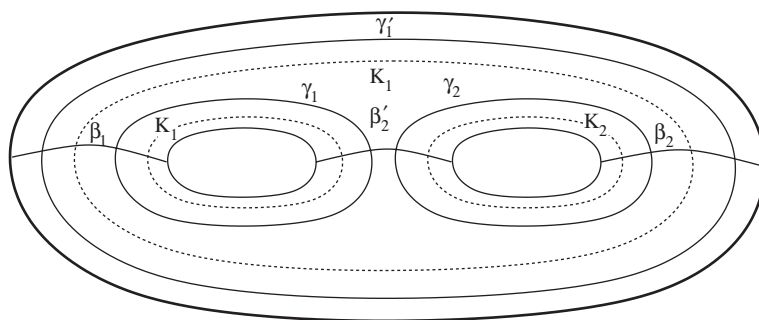


Fig. 4. Four-dimensional handleslides: An illustration of four-dimensional handleslides, in the case where the knot has two components. The handleslide takes place inside the solid torus pictured; the dashed lines K_1 and K_2 constitute the first link, and K_1' is obtained by sliding K_1 over K_2 . The darker circles (β_1 , γ_1 , γ_2 , γ_1' and β_2') take place on the boundary of the handlebody, in the Heegaard diagram. The dual to K_1' , β_1' is not pictured: it is an isotopic copy of β_1 ; similarly, γ_2' is not pictured, as it is an isotopic translate of γ_2 .

Commutativity follows from associativity, and the observation that

$$F_{\beta, \beta', \gamma'}^{\leq 0}(\Theta_{\beta, \beta'} \otimes \Theta_{\beta', \gamma'}, s_0) = \pm \Theta_{\beta, \gamma'} = \pm F_{\beta, \gamma, \gamma'}^{\leq 0}(\Theta_{\beta, \gamma} \otimes \Theta_{\gamma, \gamma'}, s_0),$$

according to the handleslide invariance of the homology groups. \square

Lemma 4.15. *The maps $\underline{E}_{W_3, s}^\circ$ are invariant under the ordering of the three-handles, and handleslides amongst them.*

Proof. Again, we verify independence of the ordering, i.e.

$$\underline{E}_{S_2, s}^\circ \circ \underline{E}_{S_1, s}^\circ = \underline{E}_{S_1, s}^\circ \circ \underline{E}_{S_2, s}^\circ.$$

To see this, we can equip Y with a Heegaard decomposition which splits off a standard decomposition of $\#^2(S^2 \times S^1)$: $(E \# E, \{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}, z)$. It is clear from the definitions now that the two composite maps agree. \square

Lemma 4.16. *Let W_1 be the cobordism obtained by adding a one-handle to Y and W_2 be the two-handle attached along any knot \mathbb{K} which cancels the one-handle. Then, the induced map*

$$\underline{F}_{\mathbb{K}, s}^\circ \circ \underline{G}_{W_1, s}^\circ : \underline{HF}^\circ(Y, s, M) \longrightarrow \underline{HF}^\circ(Y, s, M)$$

corresponds to the identity map.

Proof. Fix a Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y_1 . Let $(E, \alpha, \beta, \gamma, z_0)$ be a Heegaard triple where E is a genus one surface, α and β are exact Hamiltonian translates of one another, and γ is a curve meeting both transversally in a single intersection point. In fact, if we consider the Heegaard triple

$$(\Sigma \# E, \alpha', \beta', \gamma', z') = (\Sigma, \alpha, \beta, \gamma, z) \# (E, \alpha, \beta, \gamma, z_0),$$

where the γ are small Hamiltonian translates of the β , then this Heegaard triple represents a two-handle \mathbb{K}_0 which cancels the one-handle in $Y \times (S^2 \times S^1)$ which was just introduced. The model case calculation in E now shows that the map induced by the composite induces the same map in homology as the map

$$CF^\infty(\alpha, \beta, s) \longrightarrow CF^\infty(\alpha \cup \{\alpha\}, \beta' \cup \{\gamma\}, s)$$

given by

$$[\mathbf{x}, i] \mapsto [\mathbf{x}' \times \{c\}, i],$$

where $c \in \alpha \cap \gamma$ is the unique intersection point in E , and \mathbf{x}' is the intersection point of $\mathbb{T}_\alpha \cap \mathbb{T}'_\beta$ closest to $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$: thus, this is equivalent to the map induced by stabilization.

More generally, if we choose another two-handle \mathbb{K} which cancels the given one-handle, we claim that the induced composite is the same. This can be shown by constructing a Heegaard diagram for which the induced δ belonging to the framing of \mathbb{K} differs from γ by a sequence of isotopies and handleslides across the $\{\beta'_1, \dots, \beta'_g\}$. This follows from the fact that, if we surger out the $\{\beta'_1, \dots, \beta'_g\}$ from $\Sigma \# E$, we are left with a torus, in which γ and δ are two curves which meet the remaining β transversally in a single transverse intersection point (after isotopies, if necessary). Thus, in this torus, γ and δ are isotopic. It follows then that the two maps $\otimes \Theta_{\beta, \gamma}$ and $\otimes \Theta_{\beta, \delta}$ (where here $\delta = \{\delta_1, \dots, \delta_g, \delta\}$, and δ_i for $i = 1, \dots, g$ are small, Hamiltonian translates of the corresponding β_i) which correspond to $F^\circ_{\mathbb{K}}$ and $F^\circ_{\mathbb{K}_0}$ agree after we post-compose the second with multiply the second with $\otimes \Theta_{\delta, \gamma}$, which in turn corresponds to a strong equivalence between two Heegaard diagrams belonging to the same three-manifold $Y_1 = Y_3$. \square

Dually, we have the following:

Lemma 4.17. *Let W_1 be the cobordism obtained by attaching a two-handle to Y along a framed knot \mathbb{K} , and let W_2 be a three-handle attached along a two-sphere which cancels the knot. Then, the composite*

$$\underline{E}^\circ_{W_2, s} \circ \underline{F}^\circ_{\mathbb{K}, s} : \underline{HF}^\circ(Y, s, M) \longrightarrow \underline{HF}^\circ(Y, s, M)$$

corresponds to the identity map.

Proof. The proof follows from “turning around” the proof of Lemma 4.16. \square

Proof of Theorem 3.8. Consider the Kirby calculus picture for the cobordism W (see [8,9]). Any two such pictures can be connected by a sequence of pair cancellations and additions, and a sequence of handleslides (Kirby moves). In fact, since W is a connected manifold-with-boundary, W has such a description with no zero- or four-handles; moreover, any two such descriptions can be connected through a sequence of Kirby moves which never introduce new zero- or four-handles. On the other hand, the above lemmas ensure that the map $\underline{F}_{W,s}^\circ$ is invariant under all the Kirby moves, and hence, it is a four-manifold invariant. \square

For the untwisted case (Theorem 3.1), when we wish to construct the map $F_{W,s}^\circ$ without the action of the exterior algebra of $H_1(W; \mathbb{Z})/\text{Tors}$, we can appeal directly to Theorem 3.8 above. Specifically, suppose that W is a cobordism from Y_1 to Y_2 . Taking trivial coefficients $M = \mathbb{Z}$ for $H^1(Y_1; \mathbb{Z})$, then we have that the induced module $M(W) \cong \mathbb{Z}[K]$, where K is the kernel of the map from $H^2(W, Y_2)$ to $H^2(W)$. There is a canonical map of $\mathbb{Z}[H^1(Y_2; \mathbb{Z})]$ -modules

$$\varepsilon: \mathbb{Z}[K] \longrightarrow \mathbb{Z};$$

so we can define $F_{W,s}^\circ$ to be the composite

$$F_{W,s}^\circ = H(\varepsilon) \circ \underline{F}_{W,s}^\circ.$$

Equivalently (cf. Eq. (7)), we can construct the untwisted maps for cobordisms using the untwisted triangle construction, and observe that the proof of Theorem 3.8 (and all its lemmas) adapts with notational changes to prove Theorem 3.1. We adopt this point of view, when including the action of the one-dimensional homology.

Proof of Theorem 3.1. To construct the invariant as a map

$$F_{W,s}^\circ: HF^\circ(Y_1, t_1) \otimes \Lambda^*(H_1(W, \mathbb{Z})/\text{Tors}) \longrightarrow HF^\circ(Y_2, t_2),$$

in general, we proceed as follows.

As before, we split $W = W_1 \cup_{Y'_1} W_2 \cup_{Y'_2} W_3$, where W_1 and W_3 are collections of one- and three-handles, respectively, and W_2 is the cobordism induced by the two-handles. Let $X \subset W_2$ be the four-manifold obtained from a Heegaard triple for W_2 (i.e. X is the four-manifold underlying a Heegaard triple subordinate to some bouquet for the link \mathbb{L} describing W_2). Clearly, W is obtained from X by adding three- and four-handles, so in particular the inclusion determines an isomorphism $H_1(X) \cong H_1(W)$. Thus, we define for each $\gamma \in \Lambda^*(H_1(W)/\text{Tors}) \cong \Lambda^*(H_1(X)/\text{Tors})$,

$$F_{W,s}^\circ(\gamma \otimes \xi) = E_{V,s}^\circ \circ F_{\mathbb{L},s}^\circ(\gamma \otimes G_{W_1,s}^\circ),$$

where we now use the extended triangle map for F° using the exterior algebra on the one-dimensional homology (cf. Lemma 2.6).

The proof of Theorem 3.8 now adapts with the following observations. First, note that equivalences Ψ of Heegaard diagrams induce maps on Floer homology which are equivariant under $H_1(Y)/\text{Tors}$ -actions. Thus, we can include the action in the analogue of Theorem 4.4: if $(\Sigma_1, \alpha_1, \beta_1, \gamma_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, \gamma_2, z_2)$ are a pair of Heegaard triples subordinate to two bouquets for a link in Y , then the following diagram commutes:

$$\begin{array}{ccc} HF^\circ(\alpha_1, \beta_1, \mathfrak{s}|Y) \otimes \Lambda^*(H_1(X)/\text{Tors}) & \xrightarrow{F_{W,\mathfrak{s}}^\circ{}^1} & HF^\circ(\alpha_1, \gamma_1, \mathfrak{s}|Y(\mathbb{L})) \\ \Psi_1^\circ \downarrow & & \Psi_2^\circ \downarrow \\ HF^\circ(\alpha_2, \beta_2, \mathfrak{s}|Y) \otimes \Lambda^*(H_1(X)/\text{Tors}) & \xrightarrow{F_{W,\mathfrak{s}}^\circ{}^2} & HF^\circ(\alpha_2, \gamma_2, \mathfrak{s}|Y(\mathbb{L})) \quad \square \end{array}$$

The treatment of one- and three-handles goes through again with only notational changes. We observe that the map

$$G_{W_1,\mathfrak{s}}^\circ: HF^\circ(Y) \longrightarrow HF^\circ(Y\#(S^2 \times S^1))$$

is equivariant under the action of $H_1(Y)/\text{Tors}$, according to Proposition 6.4 of [15]. It follows easily that if W and W' differ by the addition of a canceling pair of one- and two-handles, then the induced maps $F_{W,\mathfrak{s}}^\circ$ and $F_{W',\mathfrak{s}}^\circ$, thought of as maps including the exterior algebra action, agree. The result now follows.

4.5. Finiteness

We turn now to Theorem 3.3.

Proof of Theorem 3.3. It suffices to consider the case where W consists only of two-handles. Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a subordinate Heegaard triple, which we can assume is strongly \mathfrak{s} -admissible (in the sense of Definition 8.8 of [16], see also Section 2.3 of the present paper). Note that this admissibility condition depends on \mathfrak{s} only through its restriction to the boundary. We wind further to achieve the following additional admissibility condition: we arrange that each triply periodic domain with $n_z(\mathcal{P}) = 0$ has both positive and negative coefficients. It is easy to see that under this hypothesis, for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{w} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and for each integer j , there are only finitely many homotopy classes of maps $\psi \in \pi_2(\mathbf{x}, \mathbf{w}, \mathbf{y})$ with $n_z(\psi) = j$ which support holomorphic representatives. Both finiteness conditions are now clearly satisfied. \square

4.6. Composition laws revisited

The composition laws for cobordism invariants follows readily from the definitions, together with the version of the composition law already stated for links (cf. Proposition 4.9). But first, we must verify that we can commute two-handle additions past

one-handle additions; and similarly, three-handle additions commute past two-handle additions.

Proposition 4.18. *Let $\mathbb{L} \subset Y$ be a link in a three-manifold, and let \mathbb{L}' be the corresponding link in the three-manifold $Y' = Y \# (\#^n(S^2 \times S^1))$ obtained by attaching one-handles. Then, the following diagram commutes:*

$$\begin{array}{ccc} \underline{HF}^\circ(Y, \mathfrak{s}|Y) & \xrightarrow{F_{\mathbb{L}, \mathfrak{s}}^\circ} & HF^\circ(Y(\mathbb{L}), \mathfrak{s}|Y(\mathbb{L})) \\ \underline{G}_{V, \mathfrak{s}}^\circ \downarrow & & \underline{G}_{V(\mathbb{L}), \mathfrak{s}}^\circ \downarrow \\ \underline{HF}^\circ(Y', \mathfrak{s}|Y') & \xrightarrow{F_{\mathbb{L}', \mathfrak{s}}^\circ} & HF^\circ(Y'(\mathbb{L}'), \mathfrak{s}|Y'(\mathbb{L}')) \end{array}$$

Proof. This follows immediately from the gluing result for holomorphic triangles, Theorem 2.14, together with the model case of a torus with three (homologically non-trivial, simple, closed) curves in the torus which are exact Hamiltonian translates of one another. (In fact, we have verified this proposition in a special case when we showed in Theorem 4.10 that G° commutes with the maps induced by equivalences of Heegaard diagrams; but the case where the horizontal map is induced more generally by a Heegaard triple, the proof is no different.) \square

Similarly, we have the following:

Proposition 4.19. *Let $\mathbb{L}' \subset Y'$ be a link in a three-manifold, which is disjoint from a non-separating two-sphere $S \subset Y'$, then the following diagram commutes:*

$$\begin{array}{ccc} \underline{HF}^\circ(Y', \mathfrak{s}|Y') & \xrightarrow{F_{\mathbb{L}', \mathfrak{s}}^\circ} & HF^\circ(Y'(\mathbb{L}'), \mathfrak{s}|Y'(\mathbb{L}')) \\ \underline{E}_{U, \mathfrak{s}}^\circ \downarrow & & \underline{E}_{U(\mathbb{L}), \mathfrak{s}}^\circ \downarrow \\ \underline{HF}^\circ(Y, \mathfrak{s}|Y) & \xrightarrow{F_{\mathbb{L}, \mathfrak{s}}^\circ} & HF^\circ(Y(\mathbb{L}), \mathfrak{s}|Y(\mathbb{L})) \end{array}$$

where Y is the result of adding a three-handle to Y' along S , and $\mathbb{L} \subset Y$ is the link induced from \mathbb{L}' .

Proof. We construct the Heegaard triple $(\Sigma', \alpha', \beta', \gamma', z')$ for Y' so that the last curves are α_{g+1} , β_{g+1} and γ_{g+1} are small Hamiltonian isotopic translates of one another, all supported inside a summand E of $\Sigma' = \Sigma \# E$, so that the $\alpha = \{\alpha_1, \dots, \alpha_g\}$, $\beta = \{\beta_1, \dots, \beta_g\}$ and $\gamma = \{\gamma_1, \dots, \gamma_g\}$ are supported inside Σ . Then, $(\Sigma, \alpha, \beta, \gamma, z)$ is a Heegaard triple for $\mathbb{L} \subset Y$, and the result follows from the gluing for holomorphic triangles with the model calculation of $(E, \alpha_{g+1}, \beta_{g+1}, \gamma_{g+1}, z_0)$ as before. \square

Proof of Theorem 3.9. The proof now follows easily from Propositions 4.9, 4.18, and 4.19. \square

The need to sum over Spin^c structures for the composition law in the untwisted case follows from the corresponding fact in the associativity of the triangle construction. This results in the following analogue of Proposition 4.9 in the untwisted case:

Proposition 4.20. *Suppose that W_1 and W_2 are composed entirely of two-handles, then there are choices of sign for F_{W_1, s_1}° and F_{W_2, s_2}° with the property that*

$$F_{W_2, s_2}^\circ \circ F_{W_1, s_1}^\circ = \sum_{\{s \in \text{Spin}^c(W_1 \# W_2) \mid s|_{W_1=s_1}, s|_{W_2=s_2}\}} \pm F_{W_1 \# W_2, s}^\circ.$$

More generally, given $\zeta_1 \in \Lambda^*(H_1(W_1; \mathbb{Z})/\text{Tors})$ and $\zeta_2 \in \Lambda^*(H_1(W_2, \mathbb{Z})\text{Tors})$, we have that

$$F_{W_2, s_w}^\circ(\zeta_2 \otimes F_{W_1, s_1}^\circ(\zeta_1 \otimes \cdot)) = \sum_{\{s \in \text{Spin}^c(W_1 \# W_2) \mid s|_{W_1=s_1}, s|_{W_2=s_2}\}} F_{W_1 \# W_2, s}^\circ(\zeta_3 \otimes \cdot),$$

where $\zeta_3 \in \Lambda^*(H_1(W_1 \# W_2)/\text{Tors})$ is the image of $\zeta_1 \otimes \zeta_2$ under the natural map.

Proof. Apply the proof of Proposition 4.9, only using associativity in the untwisted case. Let $(\Sigma, \alpha, \beta, \gamma, z)$ and $(\Sigma, \alpha, \gamma, \delta, z)$ be the two Heegaard triples constructed in the proof. To verify that the action of H_1 is respected as claimed, we represent the action of $\zeta_1 \in H_1(X_{\alpha, \beta, \gamma}; \mathbb{Z}) \cong H_1(W_1)$ by the action of a class from $H_1(Y_{\alpha, \beta})$, which in turn is a constraint from $H^1(\mathbb{T}_\alpha)$. It follows immediately that:

$$F_{W_2, s_w}^\circ(1 \otimes F_{W_1, s_1}^\circ(\zeta_1 \otimes \cdot)) = \sum_{\{s \in \text{Spin}^c(W_1 \# W_2) \mid s|_{W_1=s_1}, s|_{W_2=s_2}\}} F_{W_1 \# W_2, s}^\circ(\zeta_1 \otimes \cdot)$$

(where the element ζ_1 appearing on the right-hand side is to be interpreted as the image of ζ_1 inside $H_1(W_1 \# W_2; \mathbb{Z})$). Classes ζ_2 coming from $H_1(X_{\alpha, \gamma, \delta}) \cong H_1(W_2)$ can be represented by classes from $H_1(Y_{\alpha, \delta})$, to get the corresponding equation that

$$F_{W_2, s_w}^\circ(\zeta_2 \otimes F_{W_1, s_1}^\circ(1 \otimes \cdot)) = \sum_{\{s \in \text{Spin}^c(W_1 \# W_2) \mid s|_{W_1=s_1}, s|_{W_2=s_2}\}} F_{W_1 \# W_2, s}^\circ(\zeta_2 \otimes \cdot). \quad \square$$

Proof of Theorem 3.4. With Proposition 4.20 replacing Proposition 4.9, the proof of Theorem 3.9 applies. \square

5. Duality and conjugation invariance

In the present section, we describe duality for the maps, and the closely related conjugation invariance.

5.1. Duality

We discuss duality for the maps associated to cobordisms, proving Theorems 3.5 and 3.6. But first, we recall some aspects of duality for the homology groups of a three-manifold.

If Y is a three-manifold, then we can think of $\text{Spin}^c(Y)$ structures as equivalence classes of nowhere vanishing vector fields. Since a nowhere vanishing vector field over Y can also be viewed as a vector field over $-Y$, we get a naturally induced bijection

$$\text{Spin}^c(Y) \cong \text{Spin}^c(-Y).$$

In terms of Heegaard diagrams, if the diagram $(\Sigma, \alpha, \beta, z)$ describes $-Y$, then $(-\Sigma, \alpha, \beta, z)$ describes Y . Under the above identification, if $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an intersection point, then the Spin^c structure induced by \mathbf{x} and the basepoint z for the first Heegaard diagram agrees with that induced by \mathbf{x} and z for the second diagram.

We define a pairing

$$\langle, \rangle: CF^\infty(Y, \mathfrak{s}) \otimes CF^\infty(-Y, \mathfrak{s}) \longrightarrow \mathbb{Z},$$

by the formula

$$\langle [\mathbf{x}, i], [\mathbf{y}, j] \rangle = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } i + j + 1 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where we are using $(\Sigma, \alpha, \beta, z)$ for Y and $(-\Sigma, \alpha, \beta, z)$ for $-Y$ as above.

Lemma 5.1. *Under the above pairing, we have the identities:*

$$\langle \xi, \partial_{-Y}^\infty \eta \rangle = \langle \partial_Y^\infty \xi, \eta \rangle, \quad (8)$$

$$\langle \xi, U\eta \rangle = \langle U\xi, \eta \rangle. \quad (9)$$

Proof. Now, precomposition of the reflection of the disk across the real axis gives a map from $\pi_2(\mathbf{x}, \mathbf{y})$ to $\pi_2(\mathbf{y}, \mathbf{x})$ which identifies the \mathbf{j} -holomorphic maps in Σ with the $(-\mathbf{j})$ -holomorphic maps in $-\Sigma$ (indeed, if J_t is a one-parameter family of almost-complex structures over $\text{Sym}^g(\Sigma)$, then this reflection identifies the J_t -pseudo-holomorphic maps with the $(-J_t)$ -pseudo-holomorphic maps in $\text{Sym}^g(-\Sigma)$). The first formula follows. The second formula is immediate. \square

According to Eq. (8), the pairing we have defined descends to give pairings

$$\langle, \rangle: HF^\infty(Y, \mathfrak{s}) \otimes HF^\infty(-Y, \mathfrak{s}) \longrightarrow \mathbb{Z},$$

$$\langle, \rangle: HF^+(Y, \mathfrak{s}) \otimes HF^-(-Y, \mathfrak{s}) \longrightarrow \mathbb{Z}.$$

Indeed, according to Eq. (9), for this second pairing, $HF_{\text{red}}^{-}(-Y, \mathfrak{s})$ pairs trivially with the image of $HF^{\infty}(Y, \mathfrak{s})$ inside $HF_{\text{red}}^{+}(Y, \mathfrak{s})$; thus, we obtain an induced pairing

$$\langle , \rangle : HF_{\text{red}}^{+}(Y, \mathfrak{s}) \otimes HF_{\text{red}}^{-}(-Y, \mathfrak{s}) \longrightarrow \mathbb{Z}.$$

In the case where $c_1(\mathfrak{s})$ is torsion, the pairing (on the level of chain complexes) induces an isomorphism between the chain complex for $CF^{+}(Y, \mathfrak{s})$ and the cochain complex for $CF^{-}(-Y, \mathfrak{s})$.

If W is a cobordism from Y_1 to Y_2 , then, W can also be thought of as a cobordism from $-Y_2$ to $-Y_1$. We show that

$$F_{W, \mathfrak{s}}^{+} : HF^{+}(Y_1, \mathfrak{s}|Y_1) \longrightarrow HF^{+}(Y_2, \mathfrak{s}|Y_2)$$

is adjoint (under the above pairing) to the map

$$F_{W, \mathfrak{s}}^{-} : HF^{-}(-Y_2, \mathfrak{s}|Y_2) \longrightarrow HF^{-}(-Y_1, \mathfrak{s}|Y_1),$$

as stated in Theorem 3.5.

Proof of Theorem 3.5. When viewing W as a cobordism from $-Y_2$ to $-Y_1$, three-handles are viewed as one-handles, the one-handles are viewed as three-handles, and the collection of two handles are viewed as another dual collection.

In particular, we consider the case where W consists of a single one-handle, attached to $Y_1 = Y$ to obtain $Y_2 = Y \# (S^1 \times S^2)$. Suppose that Y_1 is represented by $(\Sigma, \alpha, \beta, z)$, and then that $Y_2 = (\Sigma, \alpha, \beta, z) \# (E, \alpha_0, \beta_0, z_0)$ where $(E, \alpha_0, \beta_0, z_0)$ is a standard Heegaard splitting of $S^1 \times S^2$. Then, we can alternatively think of the cobordism as going from $-Y_2 = (-\Sigma, \alpha, \beta, z) \# (-E, \alpha_0, \beta_0, z_0)$ to $(-\Sigma, \alpha, \beta, z)$ by attaching a three-handle to the two-sphere specified by the pair α_0 and β_0 . Now, as defined in Section 4.3, F_W^{+} is the map taking $[\mathbf{x}, i]$ to $[\mathbf{x} \times \{\theta\}, i]$, where θ is the intersection point of $\alpha_0 \cap \beta_0$ which has higher Maslov index. But viewed as an intersection point for $(-E, \alpha_0, \beta_0, z_0)$, θ has lower Maslov index. It follows then immediately from the definition of the map on three-handles (cf. Section 4.3), that $F_{W, \mathfrak{s}}^{+}$ as a map from the homology Y_1 to Y_2 is dual to $F_{W, \mathfrak{s}}^{-}$ as a map from the homology of $-Y_2$ to $-Y_1$. The result follows with notational changes when W is obtained as a union of one-handles (or three-handles).

We now focus on the case of two-handles. Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a Heegaard triple subordinate to some bouquet for the link $\mathbb{L} \subset Y$. Then, the Heegaard triple $(-\Sigma, \alpha, \gamma, \beta, z)$ is a Heegaard triple subordinate some link $\mathbb{L}' \subset -Y(\mathbb{L})$, on which a surgery gives $-Y$. We must compare holomorphic triangles for these two Heegaard triples.

Observe that there is a unique antiholomorphic involution R of the triangle Δ with edges α , β , and γ , which switches the β and γ edges and maps the α edge to itself. Precomposing ψ' with R induces an identification for each $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$, $\mathbf{w} \in \mathbb{T}_{\beta} \cap \mathbb{T}_{\gamma}$,

$\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ of classes

$$\pi_2(\mathbf{x}, \mathbf{w}, \mathbf{y}) \cong \pi_2(\mathbf{y}, \mathbf{w}, \mathbf{x})$$

which carries all the Σ -holomorphic to $-\Sigma$ holomorphic data: $\mu_\Sigma(\psi) = \mu_{-\Sigma}(\psi \circ R)$ and $\mathcal{M}_\Sigma(\psi) = \mathcal{M}_{-\Sigma}(\psi \circ R)$. Since R reverses orientation, we have that $n_z^\Sigma(\psi \circ R) = n_z^{-\Sigma}(\psi)$. Thus,

$$\begin{aligned} \langle F_{\mathbb{L}, \mathfrak{s}}^+([\mathbf{x}, i] \otimes [\mathbf{w}, j]), [\mathbf{y}, k] \rangle &= \sum_{\{\psi \in \pi_2(\mathbf{x}, \mathbf{w}, \mathbf{y}) \mid \mu_\Sigma(\psi)=0, \mathfrak{s}_z(\psi)=\mathfrak{s}, n_z(\psi)=i+j+k+1\}} \# \mathcal{M}_\Sigma(\psi) \\ &= \sum_{\{\psi \in \pi_2(\mathbf{y}, \mathbf{w}, \mathbf{x}) \mid \mu_{-\Sigma}(\psi)=0, \mathfrak{s}_z(\psi)=\mathfrak{s}, n_z(\psi)=i+j+k+1\}} \# \mathcal{M}_{-\Sigma}(\psi) \\ &= \langle [\mathbf{x}, i], F_{\mathbb{L}', \mathfrak{s}}^+([\mathbf{y}, k] \otimes [\mathbf{w}, j]) \rangle. \end{aligned}$$

Observe next that if Θ is some chain in $CF^{\leq 0}(\beta, \gamma, z)$ representing a generator for the highest dimensional non-trivial homology of $HF^{\leq 0}(\#^n(S^1 \times S^2), \mathfrak{s}_0)$ (where we are using the Heegaard diagram $(\Sigma, \beta, \gamma, z)$), then Θ can be viewed as a chain for $CF^{\leq 0}(\gamma, \beta, z)$ (using $-\Sigma$ as the dividing surface); as such it will also represent a generator for the highest non-trivial homology of $HF^{\leq 0}(\#^n(S^1 \times S^2), \mathfrak{s}_0)$ (this follows from the conjugation invariance of the three-dimensional groups, together with the observation that \mathfrak{s}_0 is fixed under conjugation). The theorem follows similarly when W is composed of a collection of two-handles. The general case follows, combined with the remarks on one- and three-handles at the beginning of the proof. \square

5.2. Conjugation invariance

Recall that if $(\Sigma, \alpha, \beta, z)$ is a Heegaard decomposition for Y , then $(-\Sigma, \beta, \alpha, z)$ also represents Y , and if $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an intersection point, then the two Spin^c structure associated to \mathbf{x} using the two Heegaard diagrams are conjugates of one another. This induces isomorphisms

$$\mathfrak{J}_Y^\circ: HF^\circ(Y, \mathfrak{t}) \longrightarrow HF^\circ(Y, \bar{\mathfrak{t}}).$$

We wish to prove the corresponding conjugation invariance of maps associated to cobordisms. But, first we prove a technical lemma, regarding the invariant of maps associated to framed links.

Consider a framed link $\mathbb{L} \subset Y$. We call a Heegaard triple $(\Sigma, \delta, \alpha, \beta, z)$ *left-subordinate* to a bouquet for the link if $Y_{\delta, \alpha} \cong \#^\ell(S^1 \times S^2)$, $Y_{\alpha, \beta} \cong Y$, and $Y_{\delta, \alpha} \cong Y(\mathbb{L})$. We define a link invariant

$$K_{\mathbb{L}, \mathfrak{s}}^\circ: HF^\circ(Y, \mathfrak{s}|_Y) \longrightarrow HF^\circ(Y(\mathbb{L}), \mathfrak{s}|_{Y(\mathbb{L})})$$

using a left-subordinate Heegaard triple, using the formula

$$K_{\mathbb{L},s}^{\circ}(\xi) = f^{\circ}(\Theta_{\delta,\alpha} \otimes \xi, s),$$

where $\Theta_{\delta,\alpha}$ is, once again, a top-dimensional generator of $HF^{\leq 0}(\#^{\ell}(S^2 \times S^1), s_0)$. This agrees with $F_{\mathbb{L},s}^{+}$, as defined in Section 4.1, according to the following:

Lemma 5.2. *The invariant of the link $K_{\mathbb{L},s}^{+}$ agrees with $F_{\mathbb{L},s}^{+}$ up to sign.*

Proof. For each link, we can find a Heegaard quadruple

$$(\Sigma, \delta, \alpha, \beta, \gamma, z)$$

with the property that $(\Sigma, \delta, \alpha, \beta, z)$ is left-subordinate to $\mathbb{L} \subset Y$, $(\Sigma, \alpha, \beta, \gamma, z)$ is subordinate (in the usual sense) to the link $\mathbb{L} \subset Y$, $(\Sigma, \delta, \beta, \gamma, z)$ is subordinate to a cobordism from Y to $Y\#^n(S^1 \times S^2)$ (obtained as zero-surgery on an n -component framed link), and $(\Sigma, \delta, \alpha, \gamma)$ is left-subordinate to the same cobordism. Moreover, the Heegaard diagram $(\Sigma, \delta, \gamma, z)$ is equipped with n distinguished two-spheres which can be cancelled (by adding three-handles) to obtain a Heegaard diagram for Y back. We then consider the diagram

$$\begin{array}{ccc} HF^{+}(\alpha, \beta, s|Y_{\alpha,\beta}) & \xrightarrow{f^{+}(\cdot \otimes \Theta_{\beta,\gamma}, s)} & HF^{+}(\alpha, \gamma, s|Y_{\alpha,\gamma}) \\ f^{+}(\Theta_{\delta,\alpha} \otimes \cdot, s) \downarrow & & \downarrow f^{+}(\Theta_{\delta,\alpha} \otimes \cdot, s) \\ HF^{+}(\delta, \beta, s|Y_{\delta,\beta}) & \xrightarrow{f^{+}(\cdot \otimes \Theta_{\beta,\gamma}, s)} & HF^{+}(\delta, \gamma, s|Y_{\delta,\gamma}) \end{array}$$

which commutes by the usual associativity. Post-composing with the three-handles, (which cancel the two-handles whether they are added “from the left” according to Lemma 4.17, as in the bottom row above, or “from the right” according to a suitably modified version of that lemma, as in the right column), we see that the result follows.

We have pictured the Heegaard quadruple in the case where the link has a single component, cf. Fig. 5. \square

We can now prove the conjugation invariance of the maps associated to cobordisms.

Proof of Theorem 3.6. The result is obvious for cobordisms consisting only of one- and three-handles. Thus, we focus on the case of two-handles: i.e. $W = W(\mathbb{L})$ for some framed link $\mathbb{L} \subset Y$ inside $Y = Y_1$. Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a Heegaard triple subordinate to some bouquet for the link. Then, $(-\Sigma, \gamma, \beta, \alpha)$ represents the same oriented four-manifold, and indeed, it is left-subordinate to the link $\mathbb{L} \subset Y$. We claim that for each chain $\xi \in CF^{\circ}(\alpha, \beta, s|Y)$,

$$f_{\alpha,\beta,\gamma}^{\circ}(\xi \otimes \Theta_{\beta,\gamma}) = \mathfrak{I}_{Y(\mathbb{L})} \circ f_{\gamma,\beta,\alpha}^{\circ}(\Theta_{\gamma,\beta} \otimes \mathfrak{I}_Y(\xi)).$$

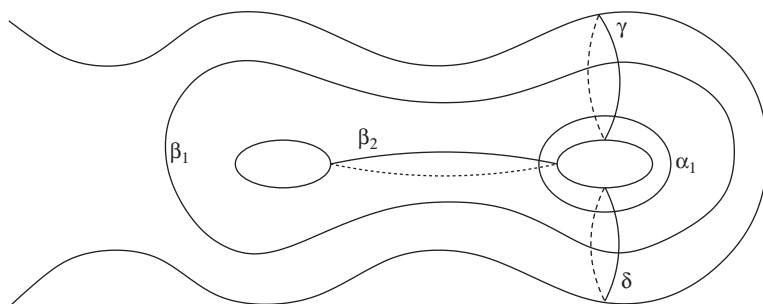


Fig. 5. Heegaard quadruple from Lemma 5.2: Let $\delta_1 = \delta$ and δ_i be a small isotopic translate of α_i for $i > 1$; let $\gamma_1 = \gamma$ and γ_i be a small isotopic translate of β_i for $i > 1$. Then, $(\Sigma, \delta, \alpha, \beta, \gamma, z)$ is a Heegaard quadruple as required by Lemma 5.2. The curves γ and δ give the two-sphere along which the three-handle can be added. (Here, our link has one component; in the more general case, we graft n copies of this picture.)

(Here, as usual, $\Theta_{\beta, \gamma}$ is a generator for the top non-zero homology group of $HF^{\leq 0}(\#^n(S^2 \times S^1), s_0)$, and so is $\mathfrak{J}_{\#^n(S^2 \times S^1)}(\Theta_{\beta, \gamma}) = \Theta_{\gamma, \beta}$.) This comes from an identification between holomorphic triangles for the two Heegaard triples which is defined by pre-composing triangles $\psi \in \pi_2(\mathbf{x}, \mathbf{w}, \mathbf{y})$ by the antiholomorphic involution of the ρ triangle which preserves the β edge and switches the α - and γ -edges.

We claim that this pre-composition induces conjugation on the level of (four-dimensional) Spin^c structures. More precisely, if

$$\Phi: X(\Sigma, \alpha, \beta, \gamma, z) \longrightarrow X(-\Sigma, \gamma\beta, \alpha, z)$$

is the obvious diffeomorphism of four-manifolds associated to Heegaard triples, then

$$\Phi^*(s_z(u \circ \rho)) \cong \mathfrak{J}(s_z(u)).$$

This can be seen on the level of the two-plane fields which determine the Spin^c structures (see [16], 8.1.4): the (singular) two-plane fields $\Phi^*(s_z(u \circ \rho))$ and $s_z(u)$ agree, only they have opposite orientations.

Conjugation invariance now follows from these observations, together with Lemma 5.2. \square

6. Blowing up

The “blow-up formula” stated in Theorem 3.7 can be reduced to a calculation in a genus one surface.

Consider a Heegaard triple $(E, \{\alpha_0\}, \{\beta_0\}, \{\gamma_0\}, z_0)$ where $\alpha_0, \beta_0, \gamma_0$ are three curves each meeting pairwise in a single intersection point, as pictured in Fig. 6 (with sub-

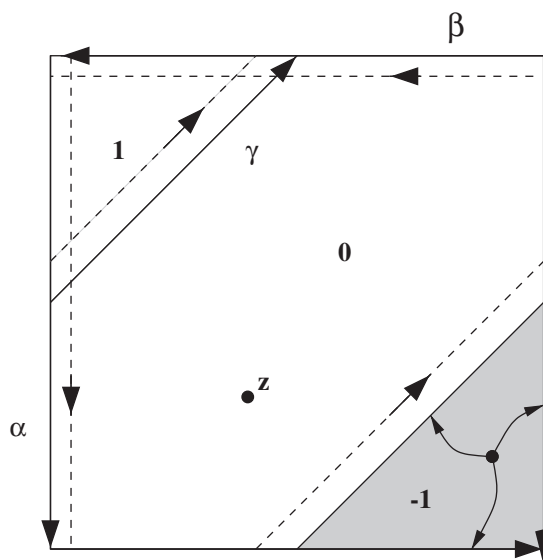


Fig. 6. Three curves in the torus: Note that the square is to be given the usual edge identifications in this picture. The dotted lines indicate the translates of the α , β , and γ curves, oriented as boundaries of the triply periodic domain whose multiplicities are shown. A dual spider is included for the small shaded triangle. Since its vertex meets the periodic domain with multiplicity -1 , and its legs are disjoint from the translated attaching circles, the dual spider number is -1 , so the evaluation $\langle c_1(\mathfrak{s}), E \rangle = +1$.

scripts dropped). This Heegaard triple is subordinate to the unknot in the three-sphere with framing -1 : i.e. the underlying four-manifold is $\overline{\mathbb{CP}}^2$ punctured in three points.

Let $x_0 = \alpha_0 \cap \beta_0$, $y_0 = \beta_0 \cap \gamma_0$, and $w_0 = \alpha_0 \cap \gamma_0$, we have that $\pi_2(x_0, y_0, w_0) \cong \mathbb{Z}$, and each homotopy class represents a different Spin^c structure \mathfrak{s} . Choosing the basepoint as in Fig. 6, we have a family of homotopy classes $\{\psi_k^\pm\}$ indexed by a sign and a positive integer k , with

$$\begin{aligned}\mu(\psi_k^\pm) &= 0, \\ n_z(\psi_k^\pm) &= \frac{k(k-1)}{2}\end{aligned}$$

(cf. Proposition 9.5 of [15]). Each homotopy class has a unique, smooth holomorphic representative, so that $\#\mathcal{M}(\psi_k^\pm) = \pm 1$.

We will use the following:

Lemma 6.1. *Let $\{\psi_k^\pm\}_{k=1}^\infty$ be the family of triangles described above, and let $E \in H_2(\overline{\mathbb{CP}}^2; \mathbb{Z})$ be a generator. Then,*

$$\left| \langle c_1(\mathfrak{s}_z(\psi_k)), [E] \rangle \right| = 2k + 1.$$

The proof is deferred to Section 6.2, after we establish a more general formula for the first Chern class of the Spin^c structure underlying a Whitney triangle, in terms of combinatorial data in a Heegaard triple (Proposition 6.3).

Proof of Theorem 3.7. The cobordism W can be expressed by a single two-handle (an unknot in Y with framing -1). Let $(\Sigma, \alpha, \beta, z)$ be a genus g Heegaard diagram for Y . Let γ be small isotopic translates of the β . Then, the connected sum of $(\Sigma, \alpha, \beta, \gamma, z)$ with $(E, \{\alpha_0\}, \{\beta_0\}, \{\gamma_0\}, z_0)$ is a Heegaard triple which is subordinate to the unknot with framing -1 . The theorem then follows immediately from the above observations, together with the gluing theorem for holomorphic triangles Theorem 2.14. \square

6.1. The first Chern class formula

We give a formula for calculating the evaluation of the first Chern class of a Spin^c structure over X , as specified by a Whitney triangle and a base-point, on a two-dimensional homology class in X , as specified by a triply periodic domain. This quantity is expressed purely in terms of data on Σ (cf. [15], for an analogous calculation in dimension three). This formula is used to establish Lemma 6.1 above.

To state the formula, we define certain quantities associated to triply periodic domains and classes in $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$: the Euler measure, and the dual spider number.

Fix a triply periodic domain \mathcal{P} , thought of as a two-chain in Σ which spans some triple $(a, b, c) \in \text{Span}([\alpha_i]_{i=1}^g) \oplus \text{Span}([\beta_i]_{i=1}^g) \oplus \text{Span}([\gamma_i]_{i=1}^g)$. We define the *Euler measure* $\widehat{\chi}(\mathcal{P})$ as follows. Find a representative

$$\Phi: F \longrightarrow \Sigma$$

representing \mathcal{P} , where F is a two-manifold with boundary, and where Φ is an immersion in a neighborhood of ∂F . The line bundle $\Phi^*(T\Sigma)$ has a canonical trivialization over ∂F , since Φ induces an isomorphism

$$TF \cong \Phi^*(T\Sigma),$$

and TF is canonically trivialized near ∂F (using the outward normal orientation on F). We define $\widehat{\chi}(\mathcal{P})$ to be the Euler number of $\Phi^*(T\Sigma)$, relative to this trivialization at ∂F ,

$$\widehat{\chi}(\mathcal{P}) = \langle c_1(\Phi^*(T\Sigma), \partial F), F \rangle.$$

Lemma 6.2. *Writing a triply periodic domain*

$$\mathcal{P} = \sum n_i \mathcal{D}_i,$$

we can calculate the Euler measure by the formula

$$\widehat{\chi}(\mathcal{P}) = \sum_i n_i \left(\chi(\text{int} \mathcal{D}) - \frac{1}{4} (\# \text{corner points of } F) \right),$$

where the corner points are to be counted with multiplicity.

Proof. Endow Σ with a Riemannian metric g for which the α , β , and γ are simultaneously geodesics, any two of which meet at right angles, and let $\Phi: F \rightarrow \Sigma$ be a branched cover representing \mathcal{P} . The relative Chern class of $\Phi^*(T\Sigma)$ is calculated by $\int_F \Phi^*(K_g)$, where K_g is the Gaussian curvature of the metric g ; which in turn can be calculated locally over each \mathcal{D} using the Gauss–Bonet formula to give the formula stated above. \square

Next, we set up the dual spider number of a Whitney triangle

$$u: \Delta \rightarrow \text{Sym}^g(\Sigma)$$

and a triply periodic domain \mathcal{P} .

Note first that the orientation of Σ , and the orientations on all the attaching circles α , β , and γ naturally induce “inward” normal vector fields to the attaching circles (i.e. if $\gamma: S^1 \rightarrow \Sigma$ is a unit speed immersed curve, this inward normal vector is given by $J \frac{d\gamma}{dt}$). Let α'_i , β'_i , and γ'_i denote copies of the corresponding attaching circles α'_i , β'_i , and γ'_i , translated slightly in these normal directions. Let α' , β' , and γ' denote the corresponding g -tuples, and \mathbb{T}'_α , \mathbb{T}'_β , and \mathbb{T}'_γ be the corresponding tori in $\text{Sym}^g(\Sigma)$. By construction, then, $u(e_\alpha)$ misses \mathbb{T}'_α , $u(e_\beta)$ misses \mathbb{T}'_β , and $u(e_\gamma)$ misses \mathbb{T}'_γ .

Let $x \in \Delta$ be a point in the interior, chosen in general position, so that the g -tuple $u(x)$ misses all of α' , β' , and γ' . Choose three paths a , b , and c from x to e_0 , e_1 , and e_2 , respectively. The central point x and the three paths a , b , and c is called a *dual spider*. We can think of the paths a , b , and c as one-chains in Σ . Recall that $\partial \mathcal{P}$ has three types of boundaries: the α , β , and γ boundaries, which we denote $\partial_\alpha \mathcal{P}$, $\partial_\beta \mathcal{P}$, and $\partial_\gamma \mathcal{P}$. Let $\partial'_\alpha \mathcal{P}$, $\partial'_\beta \mathcal{P}$, and $\partial'_\gamma \mathcal{P}$ respectively denote the one-chains obtained by translating the corresponding boundary components using the induced normal vector fields. The dual spider number of u and \mathcal{P} is defined by

$$\sigma(u, \mathcal{P}) = n_{u(x)}(\mathcal{P}) + \#(a \cap \partial'_\alpha \mathcal{P}) + \#(b \cap \partial'_\beta \mathcal{P}) + \#(c \cap \partial'_\gamma \mathcal{P}).$$

It is elementary to verify that $\sigma(u, \mathcal{P})$ depends only on the homotopy class of u and the periodic domain \mathcal{P} ; in particular, it is independent of the choice of dual spider used in its definition.

Proposition 6.3. Fix a Whitney triangle u , a base-point z , and a triply periodic domain \mathcal{P} , whose boundary represents the two-dimensional homology class $H(\mathcal{P}) \in H_2(X; \mathbb{Z})$.

Then,

$$\langle c_1(\mathfrak{s}_z(u)), H(\mathcal{P}) \rangle = \widehat{\chi}(\mathcal{P}) + \#(\partial\mathcal{P}) - 2n_z(\mathcal{P}) + 2\sigma(u, \mathcal{P}). \quad (10)$$

Proof. To calculate the first Chern class, we adopt the notation from Section 8.1.4 of [16], where the Spin^c structure belonging to a triangle is constructed. The construction is done by constructing a two-plane field \mathcal{L} in X , which is defined away from a finite collection of balls in the four-manifold X . We make use of the complex decomposition $TX \cong \mathcal{L} \oplus \mathcal{L}^\perp$ (which holds wherever \mathcal{L} is defined), so that

$$c_1(TX) \cong c_1(\mathcal{L} \oplus \mathcal{L}^\perp).$$

Consider the representative for $H(\mathcal{P})$ constructed earlier. We can assume that this representative misses the locus in X where the two-plane field \mathcal{L} is undefined. We think of its projection to Δ is a dual spider for u , with arcs a , b , and c .

Note that \mathcal{L}^\perp has a nowhere-vanishing vector field defined on the support of our representative, so its first Chern class evaluates trivially. (Such a vector field can be obtained by pulling back a vector field defined over the triangle which does not vanish on the support of the dual spider.) Thus,

$$\langle c_1(\mathfrak{s}_z(u)), H(\mathcal{P}) \rangle = \langle c_1(\mathcal{L}), H(\mathcal{P}) \rangle.$$

Now, we perform the Chern class evaluation in three parts: the region over the center point of the spider, where the representative is identified with the periodic domain \mathcal{P} , the region over the three legs a , b , and c , where the representative is given by a number of cylinders, and the regions in $U_i \times e_i$ (with $i = \alpha, \beta$, or γ), where the representative is given by a collection of disks. To justify this, recall that $\mathcal{L}|_{\partial\mathcal{P}}$ is canonically identified with the tangent bundle to F , where it has a canonical trivialization, given by the tangents along the curve. Similarly, over the other three boundary points of a , b , and c respectively, \mathcal{L} is identified with the tangent bundle to the disk, and then calculate the first Chern number of \mathcal{L} by evaluating the three corresponding relative first Chern numbers.

For the region over the central point x in the spider, we have the identification $\mathcal{L} \cong T\Sigma$, away from tubular neighborhoods of the $g + 1$ points $u(\tau)$ and z . Indeed, by a local calculation around these $g + 1$ special points (see [15], Lemma 7.6), it follows that

$$\begin{aligned} & \langle e(\mathcal{L}, \partial F), F \rangle \\ &= \langle e(\Phi^*(T\Sigma), \partial F), F \rangle + 2\#\{x \in F \mid \Phi(x) \in u(\tau)\} - 2\#\{x \in F \mid \Phi(x) = z\} \\ &= \widehat{\chi}(\mathcal{P}) + 2n_{u(\tau)}(\mathcal{P}) - 2n_z(\mathcal{P}). \end{aligned}$$

Consider, next, the cylinders and then the caps which are added to F to obtain the representative in X —these cylinders consist of some number of copies of $\alpha'_i \times a \subset \Sigma \times \Delta$,

$\beta'_i \times b \subset \Sigma \times \Delta$, and $\gamma'_i \times c \subset \Sigma \times \Delta$, followed then by some number of caps. Over the cylinder $\alpha_i \times a$, the restriction of \mathcal{L} is also canonically identified with the tangent bundle of the cylinder, except at $\tau \in a$ for which $u(\tau)$ meets α'_i . In a neighborhood of each such crossing, the tangent bundle to the cylinder and the actual two-plane field \mathcal{L} differ by a relative Chern number of 2 (this is the same as the contribution of the points $x \in F$ mapping to $u(\tau)$). The contribution of the β'_i and γ'_i works the same way. Hence, the contributions from the cylinders is

$$2\#(a \cap \alpha'_i) + 2\#(b \cap \beta'_i) + 2(c \cap \gamma'_i).$$

Finally, each closing disk contributes a relative Chern number of 1, since over each disk, \mathcal{L} is canonically identified with the tangent bundle, but the trivialization on the boundary is the one induced by the tangent to the boundary.

Adding this up, we get a contribution

$$\sum_{i=1}^g |a_i| + |b_i| + |c_i|.$$

Adding up all the contributions, we obtain Eq. (10). \square

6.2. Examples with $g = 1$

Lemma 6.1 is a straightforward application of Proposition 6.3 at this point.

Proof of Lemma 6.1. Consider the genus one Heegaard triple $(E, \{\alpha_0\}, \{\beta_0\}, \{\gamma_0\}, z_0)$ for $\overline{\mathbb{CP}}^2$ (punctured in three points) pictured in Fig. 6.

In the picture the multiplicities illustrated represent the triply periodic domain associated to an embedded two-sphere $E \subset \overline{\mathbb{CP}}^2$ with self-intersection number -1 .

According to Proposition 6.3, the shaded triangle— ψ_1^+ in the notation at the beginning of the section—represents a Spin^c structure with $\langle c_1(\mathfrak{s}), [E] \rangle = 1$. Specifically, this periodic domain has

$$\widehat{\chi}(\mathcal{P}) = \frac{1}{4} - \frac{1}{4} = 0,$$

$$\#(\partial\mathcal{P}) = 3,$$

$$n_z(\mathcal{P}) = 0,$$

$$\sigma(\psi_1^+, \mathcal{P}) = -1.$$

For ψ_k^+ , the only quantity which changes is $\sigma(\psi_k^+, \mathcal{P}) = k - 1$, which gives $\langle c_1(\psi_k^+), [E] \rangle = 2k + 1$. The formula $\langle c_1(\psi_k^-), [E] \rangle = -2k - 1$ follows symmetrically. \square

7. Absolute gradings

Let Y be an oriented three-manifold, equipped with a torsion Spin^c structure (i.e. one for which $c_1(\mathfrak{t})$ is torsion). We have seen that $HF^\circ(Y, \mathfrak{t})$ is a relatively \mathbb{Z} -graded Abelian group: it is generated by homogeneous elements \mathfrak{A} , on which there is a relative grading function

$$\text{gr}: \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{Z}.$$

Theorem 7.1. *Let \mathfrak{t} be a torsion Spin^c structure. Then, the homology groups $HF^\circ(Y, \mathfrak{t})$ can be endowed with an absolute grading*

$$\widetilde{\text{gr}}: \mathfrak{A} \longrightarrow \mathbb{Q}$$

satisfying the following properties:

- the homogeneous elements of least grading in $HF^+(S^3, \mathfrak{s}_0)$ have absolute grading zero,
- the absolute grading lifts the relative grading, in the sense that if $\xi, \eta \in \mathfrak{A}$, then

$$\text{gr}(\xi, \eta) = \widetilde{\text{gr}}(\xi) - \widetilde{\text{gr}}(\eta),$$

- the natural maps ι and π in the long exact sequence (Eq. (2)) preserve the absolute grading, while the coboundary map decreases absolute degree by one, and the U action decreases it by two,
- if W is a cobordism from Y_1 to Y_2 endowed with a Spin^c structure whose restriction \mathfrak{t}_i to Y_i is torsion for $i = 1, 2$, then

$$\widetilde{\text{gr}}(F_{W, \mathfrak{s}}(\xi)) - \widetilde{\text{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}, \quad (11)$$

where $\mathfrak{t}_i = \mathfrak{s}|_{Y_i}$ for $i = 1, 2$.

To define $\widetilde{\text{gr}}$, we present Y as a surgery on a link $\mathbb{L} \subset S^3$, so that \mathfrak{t} is the restriction of a Spin^c structure \mathfrak{s} over the induced cobordism $W(S^3, \mathbb{L})$ from S^3 to Y . Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a Heegaard triple subordinate to some bouquet for the link (in the sense of Definition 4.2), so that $Y_{\alpha, \beta} \cong S^3$, $Y_{\beta, \gamma} \cong \#^n(S^1 \times S^2)$, and $Y_{\alpha, \gamma} \cong Y$. Fix intersection points $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x}_1 \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ so that \mathbf{x}_0 resp. \mathbf{x}_1 are in the same degree as the highest non-zero generators of $\widehat{HF}(S^3, \mathfrak{t}_0)$ and $\widehat{HF}(\#^n(S^2 \times S^1), \mathfrak{t}_0)$ respectively. (We say simply that the intersection points \mathbf{x}_0 and \mathbf{x}_1 lie in the canonical degree.)

We define the absolute grading on $\widehat{CF}(Y, \mathfrak{t})$, by letting (for each $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$)

$$\widetilde{\text{gr}}(\mathbf{y}) = -\mu(\psi) + 2n_z(\psi) + \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}, \quad (12)$$

where $W = W(\mathbb{L})$, and $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ is a homotopy class whose Spin^c structure $\mathfrak{s}_z(\psi) = \mathfrak{s}$. This induces an absolute grading on $CF^\infty(Y, \mathfrak{t})$ by

$$\widetilde{\text{gr}}[\mathbf{y}, i] = 2i + \widetilde{\text{gr}}(\mathbf{y}),$$

and hence on the sub- and quotient-complexes $CF^-(Y, \mathfrak{t})$ and $CF^+(Y, \mathfrak{t})$ (so that the inclusion and projection preserve grading).

7.1. Invariance of the absolute grading

We show that the grading defined above is well-defined (depending only on the three-manifold and Spin^c structure), and satisfies the requirements of Theorem 7.1, in a sequence of steps which are reminiscent of the link invariant constructed in Section 4.1. Commutative diagrams coming from associativity are replaced by (more elementary) index statements.

The definition of the absolute grading as defined in Eq. (12) depends on a link $\mathbb{L} \subset S^3$, a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(W(\mathbb{L}))$ extending \mathfrak{t} , and a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, z)$ subordinate to some bouquet for the link.

Proposition 7.2. *The absolute grading defined in Eq. (12) is independent of the bouquet $B(\mathbb{L})$ for the link, and the subordinate Heegaard triple. Specifically, if $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \gamma_1, z_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \gamma_2, z_2)$ are a pair of Heegaard triples then the absolute grading induced by the first Heegaard triple on a homogeneous $\xi \in HF^\circ(\boldsymbol{\alpha}_1, \gamma_1, \mathfrak{t})$ agrees with the absolute grading induced by the second Heegaard triple on $\Psi(\xi) \in HF^\circ(\boldsymbol{\alpha}_2, \gamma_2, \mathfrak{t})$, where*

$$\Psi: HF^\circ(\boldsymbol{\alpha}_1, \gamma_1, \mathfrak{t}) \longrightarrow HF^\circ(\boldsymbol{\alpha}_2, \gamma_2, \mathfrak{t})$$

is the isomorphism induced by the equivalence of Heegaard diagrams.

This is shown in a sequence of steps.

Lemma 7.3. *For a fixed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, z)$, the absolute grading is independent of the choices of \mathbf{x}_0 and \mathbf{x}_1 in the canonical degree, and independent of the particular triangle $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ used in its definition.*

Proof. First, we show independence of the particular triangle: i.e. fix a Heegaard triple subordinate to some bouquet for the framed link $\mathbb{L} \subset S^3$, and intersection points $\mathbf{x}_0, \mathbf{x}_1$ as before. Suppose that $\psi, \psi' \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ are a pair of triangles with $\mathfrak{s}_z(\psi) = \mathfrak{s}_z(\psi') = \mathfrak{s}$. Then,

$$\psi' = \psi + \phi_{\alpha, \beta} + \phi_{\beta, \gamma} + \phi_{\alpha, \gamma} + \ell[S],$$

where $\phi_{\xi,\eta}$ are periodic domains for $Y_{\xi,\eta}$, and S is the generator of $\pi_2(\text{Sym}^g(\Sigma))$. By additivity of the Maslov index, the fact that the restriction of s to the three boundary components is torsion, and the fact that $\langle c_1(T\Sigma), [S] \rangle = 1$, it follows that

$$\mu(\psi') = \mu(\psi) + 2\ell.$$

Moreover, it is immediate that $n_z(\psi') = n_z(\psi) + \ell$. Since all the other terms of Eq. (12) remain unchanged, it follows that $\widetilde{\text{gr}}$ is independent of $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$. Independence of the choice of $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{x}_1 \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ follows similarly, with the observation that any two intersection points $\mathbf{x}_0, \mathbf{x}'_0$ for $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ both of which lie in the canonical degree can be connected by a Whitney disk $\phi \in \pi_2(\mathbf{x}_0, \mathbf{x}'_0)$ with $\mu(\phi) = n_z(\phi) = 0$. \square

Lemma 7.4. *The absolute grading is invariant under stabilizations of the Heegaard triple.*

Proof. We stabilize the triple $(\Sigma, \alpha, \beta, \gamma, z)$ by forming the connected sum with the standard genus one diagram $(E, \alpha_{g+1}, \beta_{g+1}, \gamma_{g+1}, z_0)$ encountered in Lemma 4.7. If $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$, then its stabilization $\psi' = \psi \# \psi_0$ is formed by splicing a standard triangle $\psi_0 \in \pi_2(x_0, y_0, w_0)$ in the genus one surface. Now, $\psi' \in \pi_2(\mathbf{x}'_0, \mathbf{x}'_1, \mathbf{y}')$ (where \mathbf{y}' is the stabilization of \mathbf{y} , and \mathbf{x}'_0 and \mathbf{x}'_1 are corresponding intersection points in the canonical degrees for the stabilized diagram) satisfies $\mu(\psi') = \mu(\psi)$ and $n_{z'}(\psi') = n_z(\psi)$. Of course, the topological terms in the definition of the absolute grading remain unchanged. Thus, $\widetilde{\text{gr}}(\mathbf{y}) = \widetilde{\text{gr}}(\mathbf{y}')$. \square

Lemma 7.5. *If $(\Sigma, \alpha, \beta, \gamma, z)$ and $(\Sigma', \alpha', \beta', \gamma', z')$ are strongly equivalent Heegaard triples, then the absolute gradings are identified.*

Proof. First, we observe that we are free to change the β by an isotopy without changing the relative grading. To see this, observe that if Ψ_t is an isotopy carrying β to β' , then we can find intersection points $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x}_1 \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{x}'_0 \in \mathbb{T}_\alpha \cap \mathbb{T}'_\beta$, $\mathbf{x}_1 \in \mathbb{T}_\beta \cap \mathbb{T}'_\gamma$ satisfying the requirements of the definition of the absolute grading, and also Whitney disks with dynamic boundary conditions $\phi_0 \in \pi_2^{\Psi_t}(\mathbf{x}'_0, \mathbf{x}_0)$ and $\phi_1 \in \pi_2^{\Psi_t}(\mathbf{x}'_1, \mathbf{x}_1)$ with $\mu(\phi_0) = \mu(\phi_1) = 0$ and $n_z(\phi_0) = n_z(\phi_1) = 0$. If $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ is a triangle representing s , then, we claim that the triangle with dynamic boundary conditions $\psi + \phi_0 + \phi_1$ is homotopic to a triangle $\psi' \in \pi_2(\mathbf{x}'_0, \mathbf{x}'_1, \mathbf{y})$ with (stationary boundary conditions) for the Heegaard triple $(\Sigma, \alpha, \beta', \gamma)$. By the homotopy invariance of the Maslov index and the local multiplicity, $\mu(\psi') = \mu(\psi)$ and $n_z(\psi') = \mu(\psi)$. A similar argument allows us to modify α and γ by isotopies.

Next, we claim that if the γ' are obtained from the γ by a sequence of handleslides and isotopies (so that $(\Sigma, \beta, \gamma', z)$ is still admissible for $\#^n(S^2 \times S^1)$, and of course, $(\Sigma, \alpha, \gamma', z)$ is still admissible for Y), then the induced gradings on Y are identified. This follows from the index analogue of the associativity of triangles. Specifically, suppose $\xi \in \widehat{HF}(\alpha, \gamma)$ is a homology class, and $\Phi(\xi)$ is its image under the strong

equivalence map. Then, there are intersection points $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and $\mathbf{y}' \in \mathbb{T}_\alpha \cap \mathbb{T}'_\gamma$ in the same degrees as ξ and $\Phi(\xi)$ respectively, and there is also a triangle $\psi_0 \in \pi_2(\mathbf{y}, \mathbf{x}_2, \mathbf{y}')$ with $n_z(\psi_0) = \mu(\psi) = 0$, where $\mathbf{x}_2 \in \mathbb{T}_\alpha \cap \mathbb{T}'_\gamma$ is an intersection point in the canonical degree. Now, juxtaposing the triangles $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ and $\psi_0 \in \pi_2(\mathbf{y}, \mathbf{x}_2, \mathbf{y}')$, we obtain a square $\pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}')$. We claim that there is also a triangle $\psi'_0 \in \pi_2(\mathbf{x}_1, \mathbf{x}'_1, \mathbf{x}_2)$ with $n_z(\psi'_0) = \mu(\psi'_0) = 0$, for some $\mathbf{x}'_1 \in \mathbb{T}_\beta \cap \mathbb{T}'_\gamma$; and then also a representative $\psi \in \pi_2(\mathbf{x}_0, \mathbf{x}'_1, \mathbf{y}')$ with $s_z(\psi) = s$. Now the juxtapositions of $\psi + \psi_0$ and $\psi' + \psi'_0$ represent the same Spin^c structure for the Heegaard quadruple so, up to adding doubly periodic domains (which leaves μ and n_z unchanged), we see that the two squares become homotopic, and hence that $\mu(\psi_0) + \mu(\psi) = \mu(\psi'_0) + \mu(\psi')$ and also $n_z(\psi_0) + n_z(\psi) = n_z(\psi'_0) + n_z(\psi')$. It follows that the $(\Sigma, \alpha, \beta, \gamma, z)$ -induced grading of \mathbf{y} (which is the grading of ξ) coincides with the $(\Sigma, \alpha, \beta, \gamma', z)$ -induced grading of \mathbf{y}' (which is the grading of $\Phi(\xi)$). We can apply the same arguments to allow handleslides and isotopies amongst α and β as well. \square

Proof of Proposition 7.2. Independence of the Heegaard triple subordinate to a given bouquet now follows immediately from Lemmas 7.4 and 7.5, in view of Lemma 4.5. In fact, independence of the bouquet follows as in Lemma 4.8. In that lemma, we have seen that difference choices of bouquet correspond to handleslides amongst the γ , and the proof of Lemma 7.5 actually shows that in fact these more general moves leave the absolute grading unchanged.

Since we have not yet established that $\widetilde{\text{gr}}$ is independent of the link and Spin^c structure, we include them in the notation, writing $\widetilde{\text{gr}}_{\mathbb{L}, s}([\mathbf{y}, i])$.

Lemma 7.6. *Let $Y = S^3(\mathbb{L}_1) = S^3(\mathbb{L}'_1)$, and let s_1 resp. s'_1 be Spin^c structures over $W(\mathbb{L}_1)$, resp $W(\mathbb{L}'_1)$ whose restrictions to Y agree (and are torsion). Then for any other link $\mathbb{L}_2 \subset S^3$ and Spin^c structure $s_2 \in \text{Spin}^c(W(Y, \mathbb{L}_2))$, we have that*

$$\widetilde{\text{gr}}_{\mathbb{L}_1 \cup \mathbb{L}_2, s_1 \cup s_2} - \widetilde{\text{gr}}_{\mathbb{L}'_1 \cup \mathbb{L}_2, s'_1 \cup s_2} = \widetilde{\text{gr}}_{\mathbb{L}_1, s_1} - \widetilde{\text{gr}}_{\mathbb{L}'_1, s'_1}.$$

Proof. This is the analogue of the composition law for the link invariant (Proposition 4.9).

Fix a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ subordinate to a link, and let $x_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x_1 \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{x}_2 \in \mathbb{T}_\gamma \cap \mathbb{T}_\delta$ be elements in the same degree as the corresponding top-dimensional non-zero homology in $HF^{\leq 0}(\#^\ell(S^1 \times S^2))$. We define

$$\widetilde{\text{gr}}^\circ(\mathbf{y}) = -\mu(\psi) + 2n_z(\psi).$$

Let $\psi_1 \in \pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ represent s_1 , and $\psi_2 \in \pi_2(\mathbf{y}, \mathbf{x}_2, \mathbf{w})$ represent s_2 . Up to addition of doubly periodic domains (which do not change the Maslov index), we can decompose the square obtained by juxtaposing ψ_1 and ψ_2 as another juxtaposition, of $\psi_3 \in \pi_2(\mathbf{x}_0, \mathbf{x}_3, \mathbf{y})$ and $\psi_4 \in \pi_2(\mathbf{x}_1, \mathbf{x}_3, \mathbf{y})$, for $\mathbf{x}_3 \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$, which we can also assume lies in the canonical degree. It follows that $-\mu(\psi_4) + 2n_z(\psi_4) = 0$. Now, since the

squares are homotopic, we get that

$$-\mu(\psi_1) + 2n_z(\psi_1) - \mu(\psi_2) + 2n_z(\psi_2) = -\mu(\psi_3) + 2n_z(\psi_3);$$

i.e.

$$\widetilde{\text{gr}}^\circ(\mathbf{y}) - \mu(\psi_2) + 2n_z(\psi_2) = \widetilde{\text{gr}}^\circ(\mathbf{w}),$$

where $\psi_2 \in \pi_2(\mathbf{y}, \mathbf{x}_2, \mathbf{w})$. As in the proof of Proposition 7.2 above, it follows that $-\mu(\psi_2) + 2n_z(\psi_2)$ depends only on the link \mathbb{L}_2 (thought of as a link in Y), the Spin^c structure \mathfrak{s}_2 , and, of course, the gradings of \mathbf{y} and \mathbf{w} . In particular, it is independent of \mathbb{L}_1 and \mathfrak{s}_1 .

The corresponding excisive property of the other term in $\widetilde{\text{gr}}$ is obvious. The lemma follows. \square

The three-sphere S^3 has a standard absolute grading for which $\widehat{HF}(S^3)$ is supported in degree zero.

Lemma 7.7. *Let $\mathbb{L} \subset S^3$ be the unknot with framing -1 . Then the induced absolute grading on $S^3 \cong S^3(\mathbb{L})$, $\widetilde{\text{gr}}_{\mathbb{L}, \mathfrak{s}}$ induced from any Spin^c structure over $W(\mathbb{L})$ agrees with the standard grading of $HF^+(S^3)$.*

Proof. A Heegaard triple subordinate to the unknot with framing -1 is the standard genus one diagram with three curves α , β and γ each pairwise intersecting in one point pictured in Fig. 6. The Spin^c structures over $W(\mathbb{L})$ are represented by triangles ψ_k^\pm indexed by a sign \pm and a non-negative integer k , as in Proposition 9.5 of [15]. These triangles have $\mu(\psi_k) = 0$ and $n_z(\psi_k) = \frac{k(k-1)}{2}$. According to Lemma 6.1, $\langle c_1(\mathfrak{s}_z(\psi_k^\pm)), [E] \rangle = \pm(2k-1)$. It follows that $\widetilde{\text{gr}}[\mathbf{y}, j] = 2j$, independent of the Spin^c structure used over the cobordism (where here \mathbf{y} is the unique intersection point between α and γ). \square

Lemma 7.8. *The grading on Y induced by a link $\mathbb{L} \subset S^3$ and a Spin^c structure $\mathfrak{s} \in W(\mathbb{L})$ is invariant under handleslides between components of the link.*

Proof. As in Lemma 4.14, if \mathbb{L} and \mathbb{L}' differ by four-dimensional handleslides, then we can find Heegaard triples $(\Sigma, \alpha, \beta, \gamma, z)$ and $(\Sigma, \alpha', \beta', \gamma', z)$ subordinate to a bouquet for \mathbb{L} and \mathbb{L}' , with the property that the α' (resp. β' resp. γ') are gotten by the α (resp. β resp. γ) by a sequence of handleslides and isotopies. The fact that the gradings remain identified now follows from the proof of Lemma 7.5. \square

Lemma 7.9. *Let $\mathbb{L} \subset S^3$ be the unknot with framing $+1$ (observe the sign here). Then the induced absolute grading on $S^3 \cong S^3(\mathbb{L})$, $\widetilde{\text{gr}}_{\mathbb{L}, \mathfrak{s}}$ induced from any Spin^c structure over $W(\mathbb{L})$ agrees with the standard grading of $HF^+(S^3)$.*

Proof. Let $W(\mathbb{L})$ be the cobordism, and let $H \in H_2(W(\mathbb{L}); \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Fix a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(W(\mathbb{L}))$ with $\langle c_1(\mathfrak{s}), H \rangle = 2k + 1$. Let \mathbb{L}' be the new link obtained from \mathbb{L} by adding another unknot, this one with framing -1 , and let $E \in H_2(W(\mathbb{L}'); \mathbb{Z})$ be the new homology class introduced by this unknot. We endow $W(\mathbb{L}')$ with the Spin^c structure \mathfrak{s}' with $\langle c_1(\mathfrak{s}'), E \rangle = -2k - 1$. After handlesliding the circle with framing $+1$ over the circle with framing -1 , we obtain a new link \mathbb{L}'' consisting of a pair of circles with linking number one and framings 0 and -1 . Thus, the cobordism decomposes along $S^1 \times S^2$. It is easy to see that $c_1(\mathfrak{s}')|_{S^1 \times S^2} = 0$.

We claim that the induced absolute grading on S^3 induced from \mathbb{L}'' is the standard grading. This follows from the fact that for \mathbb{L}'' , we have that the induced grading by $2i - \mu(\psi) + 2n_z(\psi)$ is shifted up by one (this is the shift appearing in the composite cobordism $S^3 \Rightarrow (S^1 \times S^2) \Rightarrow S^3$, appearing in the surgery long exact sequence for the unknot—observe also that the relative grading here is an integer, not an integer modulo two, since we are factoring through a torsion Spin^c structure on $S^1 \times S^2$). On the other hand, it is easy to see that $c_1(\mathfrak{s}') = 0$, so

$$\frac{c_1(\mathfrak{s}')^2 - 2\chi(W(\mathbb{L}'')) - 3\sigma(W(\mathbb{L}''))}{4} = -1.$$

Since the absolute grading induced by a link is invariant under handleslides (Lemma 7.8), it follows that the induced grading on S^3 by \mathbb{L}' is also the standard grading. Finally by Lemmas 7.6 and 7.7, it follows that the absolute grading induced from the \mathbb{L} must be standard, as well. \square

Lemma 7.10. *Let $\mathbb{L} \subset S^3$ be any link with the property that $S^3(\mathbb{L}) \cong S^3$. Then for any Spin^c structure \mathfrak{s} over $W(\mathbb{L})$, the induced absolute grading on S^3 is the standard grading.*

Proof. According to a theorem of Kirby (see [9]), any two links \mathbb{L} and \mathbb{L}' which give rise to S^3 can be connected by a sequence of moves which either: introduce a new, disjoint unknot with framing ± 1 , delete a disjoint unknot with framing ± 1 , (four-dimensional) handleslides amongst the components of \mathbb{L} , and isotopies of the components of \mathbb{L} . According to Lemmas 7.7–7.9, it follows that absolute grading is unchanged under all of these operations. \square

Proof of Theorem 7.1. We put the above results together to show that the absolute grading we have defined is independent of the link and Spin^c structure used in its definition. Suppose that \mathbb{L}_1 and \mathbb{L}'_1 are a pair of framed links in S^3 so that $S^3(\mathbb{L}_1) \cong S^3(\mathbb{L}'_1) \cong Y$, equipped with Spin^c structures \mathfrak{s}_1 and \mathfrak{s}'_1 with $\mathfrak{s}_1|_Y \cong \mathfrak{s}'_1|_Y$. Then, we fix a link $\mathbb{L}_2 \subset Y$ with $Y(\mathbb{L}_2) \cong S^3$, and \mathfrak{s}_2 so that $\mathfrak{s}_2|_Y \cong \mathfrak{t}$. Lemma 7.10, the gradings on $HF^+(S^3)$ induced by the links $\mathbb{L}_1 \cup \mathbb{L}_2$ (endowed with a Spin^c structure \mathfrak{s} whose restriction to $W(\mathbb{L}_1) = \mathfrak{s}_1$, and whose restriction to $W(Y, \mathbb{L}_2)$ agrees with \mathfrak{s}_2) and $\mathbb{L}'_1 \cup \mathbb{L}_2$ (endowed with a Spin^c structure whose restriction to $W(\mathbb{L}_1) = \mathfrak{s}_1$, and whose restriction to $W(Y, \mathbb{L}_2)$ agrees with \mathfrak{s}_2) coincide. The result then follows directly from Lemma 7.6.

The fact that the absolute grading is a lift of the relative grading is a direct consequence of the additivity of the Maslov index and n_z of a triangle under juxtaposition with Whitney disks. In particular, the fact that the H_1 action decreases degree by one and U decreases degree by two follows. Moreover, the degree is by definition preserved by ι and π , and is compatible with the usual grading of $HF^+(S^3, \mathfrak{s}_0)$.

It remains to verify Eq. (11). For cobordisms composed of two-handles entirely, this equation follows from another application of the index analogue of the associativity. Next, let U be a cobordism from Y to $Y\#(S^2 \times S^1)$ consisting of a single one-handle, and equipped with the Spin^c structure $\widehat{\mathfrak{t}}$ whose restriction to Y is \mathfrak{t} . We claim that if $\mathbb{K} \subset Y$ is an unknot and W is the cobordism from Y to $Y\#(S^2 \times S^1)$ obtained by zero-surgery on \mathbb{K} , then we claim that

$$\pm \gamma \circ F_{U, \widehat{\mathfrak{t}}}^\circ = F_{W, \mathfrak{s}}^\circ,$$

where γ denotes the action of a generator $H_1(Y\#(S^2 \times S^1); \mathbb{Z})$ coming from the S^1 factor, and \mathfrak{s} is the Spin^c structure over W which with $c_1(\mathfrak{s}) = 0$, and $\mathfrak{s}|_Y = \mathfrak{t}$. This is an immediate consequence of the Heegaard diagrams (note that F_U° is the map $G_{U, \mathfrak{s}}^\circ$ as defined in Section 4.3). Similarly, if V is the cobordism from $Y\#(S^2 \times S^1)$ to Y consisting of a single three-handle addition (along a sphere S from the $S^2 \times S^1$ factor), and W' is the cobordism from $Y\#(S^2 \times S^1)$ consisting of a two-handle which cancels the two-sphere, then

$$\pm F_{V, \widehat{\mathfrak{t}}} \circ \gamma = F_{W', \mathfrak{s}}$$

for the obvious choices of Spin^c structure over V and W' (again, observe that $F_{V, \widehat{\mathfrak{t}}}$ is the map $E_{V, \widehat{\mathfrak{t}}}^\circ$ from Section 4.3). It is an easy consequence of these observations (and Eq. (11), which we already know holds for the cobordisms W and W') that both F_V and F_U increase degree by $\frac{1}{2}$, verifying Eq. (11) for one- and three-handle additions. Thus, the equation follows in general. \square

7.2. Absolute gradings and duality

When \mathfrak{t} is a torsion Spin^c structure, there is a duality map (an isomorphism):

$$\mathcal{D}: HF^+(Y, \mathfrak{t}) \longrightarrow HF_-(Y, \mathfrak{t})$$

which, on the chain level, is defined by $\mathcal{D}[\mathbf{x}, i] = \langle [\mathbf{x}, i], \cdot \rangle$ (in the notation of Section 5); i.e. $\mathcal{D}[\mathbf{x}, i] = -[\mathbf{x}, -i - 1]^*$. This sets up an isomorphism of relatively graded groups. Indeed, we have the following absolutely graded version:

Proposition 7.11. *The map*

$$\mathcal{D}: HF_i^+(Y, \mathfrak{t}) \longrightarrow HF_{-i-2}^-(Y, \mathfrak{t})$$

is an isomorphism.

Proof. The following follows from the fact that the absolute grading on S^3 is well-defined.

Let $S^3 = Y(\mathbb{L})$ for some link $\mathbb{L} \subset Y$, and $(\Sigma, \alpha, \gamma, \beta, z)$ be a corresponding Heegaard triple. Let $\psi \in \pi_2(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_0)$, with $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, $\mathbf{x}_1 \in \mathbb{T}_\gamma \cap \mathbb{T}_\beta$, and $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ (where \mathbf{x}_0 and \mathbf{x}_1 are as before). Then

$$\tilde{\text{gr}}(\mathbf{y}) = \mu(\psi) - 2n_z(\psi) - \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

Now, if $(\Sigma, \alpha, \beta, \gamma, z)$ is subordinate to a bouquet for a link $\mathbb{L} \subset S^3$ (hence representing a cobordism from S^3 to Y), then

$$(-\Sigma, \alpha, \gamma, \beta, z)$$

represents a cobordism (with only two-handles) from $-Y$ to $-S^3$. We have an identification

$$\pi_2(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}) \cong \pi_2(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_0)$$

(where the first group is taken with respect to the first Heegaard triple, the second to the second Heegaard triple) by precomposition with the reflection R on the triangle which fixes one vertex and switches the other two. Observe that

$$\mu_\Sigma(\psi) = \mu_{-\Sigma}(\psi \circ R)$$

and

$$n_z^\Sigma(\psi) = n_z^{-\Sigma}(\psi \circ R).$$

It follows that $\tilde{\text{gr}}_Y(\mathbf{y}) = -\tilde{\text{gr}}_{-Y}(\mathbf{y})$.

Thus,

$$\tilde{\text{gr}}_{-Y}\mathcal{D}[\mathbf{x}, i] = \tilde{\text{gr}}_{-Y}[\mathbf{x}, -i - 1]^* = -2i - 2 - \tilde{\text{gr}}(\mathbf{x}) = -\tilde{\text{gr}}_Y([\mathbf{x}, i]) - 2. \quad \square$$

8. Mixed invariants

Let W be a cobordism with $b_2^+(W) > 1$, then the cobordism has a refined invariant

$$F_{W, \mathfrak{s}}^{\text{mix}}: HF^-(Y_1, \mathfrak{s}|Y_1) \longrightarrow HF^+(Y_2, \mathfrak{s}|Y_2).$$

To define this, we need the following basic results:

Lemma 8.1. *Let $K \subset Y$ be a null-homologous knot in an oriented three-manifold. For any integer n , and let Y_n denote the three-manifold obtained by n -framed surgery on Y along K . We claim that for any positive integer n , the map induced on HF^∞ by the two-handle cobordism W from Y to Y_n is trivial. (Note that $b_2^+(W) = 1$.)*

Proof. We prove that with twisted coefficients, the map

$$F_{W,\mathfrak{s}}^\infty: \underline{HF}^\infty(Y, \mathfrak{s}|Y) \longrightarrow \underline{HF}^\infty(Y_n, \mathfrak{s}|Y_n)$$

is trivial, and hence, by the universal coefficients spectral sequence, it follows that for any module M over $\mathbb{Z}[H^1(Y; \mathbb{Z})]$, the induced map

$$F_{W,\mathfrak{s}}^\infty: \underline{HF}^\infty(Y, \mathfrak{s}|Y, M) \longrightarrow \underline{HF}^\infty(Y_n, \mathfrak{s}|Y_n, M)$$

(and, in particular, the untwisted version) is trivial.

Recall that in Theorem 10.12 of [15], it is shown that $\underline{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}[U, U^{-1}]$ as a module over $\mathbb{Z}[U, U^{-1}]$, with action of $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ depending on \mathfrak{s} . This endows Floer homology with an absolute $\mathbb{Z}/2\mathbb{Z}$ grading, with the convention that $\underline{HF}^\infty(Y, \mathfrak{s})$ is supported in even degrees (see [15], Section 10.4). We will prove that the map induced by W reverses this $\mathbb{Z}/2\mathbb{Z}$ grading, and hence must be trivial on \underline{HF}^∞ .

We can reverse the orientation of W to get a cobordism $-W$, thought of as a cobordism from Y_n to Y . We claim first that the map $F_{-W,\mathfrak{s}}^\infty: \underline{HF}^\infty(Y_n, \mathfrak{s}|Y_n) \longrightarrow \underline{HF}^\infty(Y, \mathfrak{s}|Y)$ is an injection. This we see by showing that it induces non-trivial maps

$$F_{-W,\mathfrak{s}}^\infty: \underline{HF}^\infty(Y_n, \mathfrak{s}|Y_n; \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{HF}^\infty(Y, \mathfrak{s}|Y; \mathbb{Z}/p\mathbb{Z})$$

for all primes $p \in \mathbb{Z}$. To this end, consider the twisted long exact sequence

$$\begin{aligned} \dots & \xrightarrow{F_{-W}^+} \underline{HF}^+(Y; \mathbb{Z}/p\mathbb{Z})[T, T^{-1}] \longrightarrow \underline{HF}^+(Y_0; \mathbb{Z}/p\mathbb{Z}) \\ & \longrightarrow \underline{HF}^+(Y_n; \mathbb{Z}/p\mathbb{Z})[T, T^{-1}] \dots \end{aligned}$$

([15], Theorem 9.23). If the map $F_{-W,\mathfrak{s}}^\infty$ were trivial for each \mathfrak{s} , then it would follow that $\underline{HF}^+(Y_0)$ contains an infinite direct sum of modules isomorphic to $\mathbb{Z}/p\mathbb{Z}[T, T^{-1}]$ (where here T acts by some element of $H^1(Y_0; \mathbb{Z})$), and hence so does the image of $\underline{HF}^\infty(Y_0)$ in $\underline{HF}^+(Y_0)$; but according to Theorem 10.12 of [15], the action of $H^1(Y_0; \mathbb{Z})$ on this module is trivial (since the image of $\underline{HF}^\infty(Y, \mathfrak{s})$ inside $\underline{HF}^+(Y, \mathfrak{s})$ is trivial when $c_1(\mathfrak{s}) \neq 0$, it follows that the image of $\underline{HF}^\infty(Y)$ in $\underline{HF}^+(Y)$ admits a trivial action by $H^1(Y; \mathbb{Z})$).

Consider now the composite cobordism $W \cup_{Y_n} (-W)$, thought of as a cobordism from Y to Y . Of course, this is obtained by two two-handle additions. By reversing the order of the two two-handles, we can decompose the cobordism as $U \cup_{Y \# (S^2 \times S^1)} \cup V$.

We claim that F_U^∞ preserves the $\mathbb{Z}/2\mathbb{Z}$ grading, while F_V^∞ reverses it, and hence, F_W^∞ must also reverse $\mathbb{Z}/2\mathbb{Z}$ grading. To verify that F_U^∞ reverses $\mathbb{Z}/2\mathbb{Z}$ grading, note that

$$\underline{HF}^\infty(Y\#(S^2 \times S^1), \mathbb{Z}[H^1(Y; \mathbb{Z})]) \cong \mathbb{Z}[U, U^{-1}] \otimes \mathbb{Z}[\gamma]/\gamma^2,$$

where here multiplication by γ corresponds to the action by a generator $\mathbb{Z} \cong H_1(S^2 \times S^1) \subset H_1(Y\#(S^2 \times S^1))$ (cf. [15], Proposition 6.4). Here, the kernel of γ corresponds to the even part of $\underline{HF}^\infty(Y\#(S^2 \times S^1))$, while the complementary subspace corresponds to the odd part. Now, F_U^∞ maps into the kernel of γ , and hence F_U^∞ preserves $\mathbb{Z}/2\mathbb{Z}$ degree, while F_V^∞ is trivial on the kernel of γ (though it is a non-trivial map), and hence it reverses $\mathbb{Z}/2\mathbb{Z}$ grading. \square

More generally, we have the following:

Lemma 8.2. *Let W be a cobordism with $b_2^+(W) > 0$. Then the induced map*

$$F_{W, \mathfrak{s}}^\infty: HF^\infty(Y_1) \longrightarrow HF^\infty(Y_2)$$

vanishes. Indeed, for any module M over $\mathbb{Z}[H^1(Y_1; \mathbb{Z})]$, the map $F_{W, \mathfrak{s}}^\infty$ vanishes as well.

Proof. We can find an embedded surface $\Sigma \subset W$ with $\Sigma \cdot \Sigma > 0$. Let M be the boundary of the tubular neighborhood of Σ . By fixing a path joining Y_1 to Σ , and taking a regular neighborhood, we break the cobordism apart into a piece from Y_1 to $Y_1\#Q$, and then from $Y_1\#Q$ to Y_2 , where here Q is the boundary of a tubular neighborhood of Σ . Since $\delta H^1(Y_1\#Q) \subset H^2(W)$ is trivial, the map $F_{W, \mathfrak{s}}^\infty$ factors through the map induced by the standard cobordism from Y_1 to $Y_1\#Q$. Thus, the result follows once we show that the map on HF^∞ vanishes for this cobordism.

We can further break the cobordism from Y_1 to $Y_1\#Q$ to cobordism from Y_1 to $Y'_1 = Y_1\#^{2g}(S^2 \times S^1)$ followed by a single two-handle addition W' , to give $Y'_2 = Y_1\#Q$, along a null-homologous knot, with positive framing. The previous lemma applies to show that the induced map on HF^∞ is trivial (indeed, for any possibly twisting). The result then follows from the composition law (Theorem 3.4). \square

Definition 8.3. Fix a Spin^c structure \mathfrak{s} over W . An *admissible cut* of a cobordism W is a three-manifold $N \subset W$ which divides W into two pieces W_1 and W_2 with $b_2^+(W_1), b_2^+(W_2) > 0$, and $\delta H^1(N; \mathbb{Z}) = 0$ in $H^2(W, \partial W; \mathbb{Z})$.

Example 8.4. Suppose that $b_2^+(W) > 1$. Let $\Sigma \subset W$ be an embedded surface with $\Sigma \cdot \Sigma > 0$, and let Q denote the boundary of its tubular neighborhood. Then, both $Y_1\#Q$ and $Q\#Y_2$ determine admissible cuts for W .

Thanks to Lemma 8.2, the image of the map

$$F_{W_1, \mathfrak{s}|_{W_1}}^-: HF^-(Y_1, \mathfrak{s}|_{Y_1}) \longrightarrow HF^-(N, \mathfrak{s}|_N)$$

lies in $HF_{\text{red}}^-(N, \mathfrak{s}|N)$. Moreover (by the same lemma), the map

$$F_{W_2, \mathfrak{s}|W_2}^+ : HF^+(N, \mathfrak{s}|N) \longrightarrow HF^+(Y_2, \mathfrak{s}|Y_2)$$

factors through the projection of $HF^+(N, \mathfrak{s}|N)$ to $HF_{\text{red}}^+(N, \mathfrak{s}|N)$. Thus, we can define

$$F_{W, N, \mathfrak{s}}^{\text{mix}} : HF^-(Y_1, \mathfrak{s}|Y_1) \longrightarrow HF^+(Y_2, \mathfrak{s}|Y_2)$$

to be the composite:

$$F_{W_2, \mathfrak{s}|W_2}^+ \circ \tau^{-1} \circ F_{W_1, \mathfrak{s}|W_1}^-$$

where

$$\tau : HF_{\text{red}}^+(N, \mathfrak{s}|N) \longrightarrow HF_{\text{red}}^-(N, \mathfrak{s}|N)$$

is the natural isomorphism induced by the coboundary map for the exact sequence of Eq. (2). In the case where $HF^+(Y_2, \mathfrak{s}|Y_2)$ has no torsion (over \mathbb{Z}), we can extend this in the natural way to

$$F_{W, N, \mathfrak{s}}^{\text{mix}} : HF^-(Y_1, \mathfrak{s}|Y_1) \otimes_{\mathbb{Z}} \Lambda^*(H_1(W; \mathbb{Z})/\text{Tors}(W)) \longrightarrow HF^+(Y_2, \mathfrak{s}|Y_2)$$

Theorem 8.5. *Let W be a cobordism with $b_2^+(W) > 1$, and let N and N' be a pair of admissible cuts. Then the mixed invariants defined by these cuts coincide.*

In the proof of this result, we will make repeated use of the following:

Lemma 8.6. *If we have a pair of admissible cuts $N, N' \subset W$ which are disjoint, then the invariants $F_{W, N, \mathfrak{s}}^{\text{mix}}$ and $F_{W, N', \mathfrak{s}}^{\text{mix}}$ agree.*

Proof. This follows from the following commutative diagram

$$\begin{array}{ccccc} HF_{\text{red}}^+(N, \mathfrak{s}|N) & \longrightarrow & HF_{\text{red}}^+(N', \mathfrak{s}|N') & \longrightarrow & HF^+(Y_2, \mathfrak{s}|Y_2) \\ \tau_1 \downarrow & & \tau_2 \downarrow & & \\ HF^-(Y_1, \mathfrak{s}|Y_1) & \longrightarrow & HF_{\text{red}}^-(N, \mathfrak{s}|N) & \longrightarrow & HF_{\text{red}}^-(N', \mathfrak{s}|N'), \end{array} \quad (13)$$

which in turn follows from the naturality of the long exact sequences of Theorem 3.1 (see Remark 3.2). \square

Proof of Theorem 8.5. We will repeatedly make use of the following trick: if $c_1, c_2 \in H_2(X; \mathbb{Z})$ are a pair of homology classes with $c_1 \cdot c_2 = 0$, then there are smoothly embedded representatives Σ_1 and Σ_2 which are disjoint. To find these, take any pair

of smoothly embedded representatives Σ_1 and Σ_0 for c_1 and c_2 , respectively, chosen so that they meet transversally. Since $c_1 \cdot c_2 = 0$, the intersection points between Σ_1 and Σ_0 come in canceling pairs. Σ_2 is obtained from Σ_0 by surgering out the pairs of canceling intersection points, and attaching cylinders which are disjoint from Σ_1 (we can find such cylinders in a small tubular neighborhood of Σ_1).

Now, when $b_2^+(W) \geq 3$, we proceed as follows. Consider cuts N and N' which divide W into $W_1 \cup_N W_2$ and $W'_1 \cup_{N'} W'_2$, respectively. Let $\Sigma_1 \subset W_1$ and $\Sigma_2 \in W'_1$ be a pair of disjoint surfaces with positive square. Since $b_2^+(W) \geq 3$, it follows that $b_2^+(W - \Sigma_1 - \Sigma_2) \geq 1$, so we can find another surface $\Sigma_3 \in W - \Sigma_1 \cup \Sigma_2$ with positive square. Let Q_1 , Q_2 , and Q_3 be the boundaries of the tubular neighborhoods. Then, by Lemma 8.6 $Y_1 \# Q_1$ and $Y_1 \# Q_2$ give cuts calculating $F_{W, N, \mathfrak{s}}^{\text{mix}}$ and $F_{W, N', \mathfrak{s}}^{\text{mix}}$, respectively. Applying the lemma once more, we see that both of these invariants agree with the invariant calculated using the cut $Q_3 \# Y_2$.

When $b_2^+(W) = 2$, we make use of the blowup formula, as follows. Fix cuts Q_1 and Q_2 and corresponding surfaces $\Sigma_1 \subset W_1$ and $\Sigma_2 \subset W'_1$ as above. Blowing up near Y_2 , the blowup formula gives

$$F_{W \# \overline{\mathbb{CP}}^2, N, \widehat{\mathfrak{s}}}^{\text{mix}} = F_{W, N, \mathfrak{s}}^{\text{mix}} \quad \text{and} \quad F_{W \# \overline{\mathbb{CP}}^2, N', \widehat{\mathfrak{s}}}^{\text{mix}} = F_{W, N', \mathfrak{s}}^{\text{mix}},$$

where $\widehat{\mathfrak{s}}$ is the Spin^c structure over $W \# \overline{\mathbb{CP}}^2$ which agrees with \mathfrak{s} in a complement of $\overline{\mathbb{CP}}^2$, and whose first Chern class $c_1(\widehat{\mathfrak{s}})$ evaluates as $+1$ on an exceptional sphere E inside $\overline{\mathbb{CP}}^2$. We can assume that $\Sigma_1 \cdot \Sigma_2 \geq 0$. In the family $n([\Sigma_1] + [\Sigma_2]) + kE$, we can find a homology class w with

$$\begin{aligned} w^2 &> 0, \\ [\Sigma_1]^2 + w^2 - ([\Sigma_1] \cdot w)^2 &< 0, \\ [\Sigma_2]^2 + w^2 - ([\Sigma_2] \cdot w)^2 &< 0. \end{aligned}$$

Let Σ_3 be an embedded surface representing w . The above conditions ensure that $b_2^+(W \# \overline{\mathbb{CP}}^2 - \Sigma_1 - \Sigma_3) = 1$, and $b_2^+(W \# \overline{\mathbb{CP}}^2 - \Sigma_2 - \Sigma_3) = 1$. Applying Lemma 8.6, it follows that the invariants of $W \# \overline{\mathbb{CP}}^2$ calculated using the cuts corresponding to Σ_1 and Σ_2 agree with that calculated using Σ_3 . Thus, the result follows. \square

In view of Theorem 8.5, we will drop the cut for the notation of the mixed invariant. We also have the following result:

Proposition 8.7. *Let W be a cobordism from Y_1 to Y_2 with $b_2^+(W) > 1$, and fix Spin^c structures t_1 and t_2 over Y_1 and Y_2 , respectively. Then, there are only finitely many Spin^c structures over W for which $F_{W, \mathfrak{s}}^{\text{mix}}$ is non-trivial.*

Proof. This follows from Theorem 3.3. Fix an admissible cut $W = W_1 \#_N W_2$. Since the kernel of the map from $HF^-(Y)$ to $HF^\infty(Y)$ (where we add up all Spin^c structures) is

$HF_{\text{red}}^-(Y) \cong HF_{\text{red}}^+(Y)$, which is a finitely generated \mathbb{Z} -module, we can find an integer d large enough that $U^d HF^-(N, t)$ injects into $HF^\infty(N, t)$ for every Spin^c structure. Let ξ_1, \dots, ξ_n be the generators for $HF^-(Y_1, t_1)$ as a $\mathbb{Z}[U]$ module. According to the finiteness theorem, there is a finite subset $\mathfrak{S}_2 \subset \text{Spin}^c(W_2)$ with the property that for $s_2 \in \mathfrak{S}_2$, the map F_{W_2, s_2}^+ is non-trivial. Moreover, according to the same theorem, there is a finite subset $\mathfrak{S}_1 \subset \text{Spin}^c(W_1)$ consisting of elements $s_1 \in \mathfrak{S}_1$ for which there is some element ξ_i with $F_{W_1, s_1}^-(\xi_i) \notin U^d HF^-(N, s|N)$. Of course, if s is any Spin^c structure over W whose restriction to W_1 is not in \mathfrak{S}_1 , the composite

$$HF^-(Y_1, s|Y_1) \xrightarrow{F_{W_1, s|W_1}^-} HF^-(N, s|N) \longrightarrow HF_{\text{red}}^-(N, s|N)$$

is trivial, so that $F_{W, s}^{\text{mix}}$ is trivial. Thus, if $F_{W, s}^{\text{mix}}$ is non-trivial, its restrictions to W_1 and W_2 are constrained to lie in the finite sets \mathfrak{S}_1 and \mathfrak{S}_2 , respectively. Since $\delta H^1(N; \mathbb{Z}) = 0$, the Spin^c structure is uniquely determined by its restrictions to W_1 and W_2 . \square

8.1. Other cuts

When we drop the hypothesis that $\delta H^1(N; \mathbb{Z}) \subset H^2(W; \mathbb{Z})$ is trivial, then we can still get information about sums of mixed invariants.

Now we have the following formula for a sum of invariants:

Proposition 8.8. *Suppose that W is separated by N into a pair of cobordisms $W_1 \cup_N W_2$ with $b_2^+(W_i) > 0$. Then, we can still form the composite*

$$F_{W_2, s|W_2}^+ \circ \tau^{-1} \circ F_{W_1, s|W_1}^-,$$

where

$$\tau: HF_{\text{red}}^+(N, s|N) \longrightarrow HF_{\text{red}}^-(N, s|N)$$

is the map as before. The composite can be expressed as a sum:

$$F_{W_2, s|W_2}^+ \circ \tau^{-1} \circ F_{W_1, s|W_1}^- = \sum_{\{s \in \text{Spin}^c(W) \mid s|W_1 = s_1, s|W_2 = s_2\}} F_{W, s}^{\text{mix}}.$$

Proof. Since $b_2^+(W_i) > 0$, the maps on HF^∞ induced by these cobordisms vanish, according to Lemma 8.2, we can find $F_{W_2, s|W_2}^+ \circ \tau^{-1} \circ F_{W_1, s|W_1}^-$. We find a further subdivision of $W_1 = W_0 \cup_{N'} W_1'$ so that N' is an admissible cut for W . This can be done by letting W_0 be a neighborhood of an embedded surface in W_1 with positive square, and W_1' be its complement. The result then follows from the arguments from Commutative Diagram (13), together with the composition law, Theorem 3.4. \square

9. Closed four-manifold invariants

Let X be a closed four-manifold with $b_2^+(X) \geq 2$. Then, by deleting a pair of disjoint balls from X , we can view it as a cobordism W from S^3 to S^3 .

We can now define the *absolute invariant of X* to be the map

$$\Phi_{X,s}: \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{Tors}) \longrightarrow \mathbb{Z}/\pm 1$$

given by $\Phi_{X,s}(U^n \otimes \zeta)$ is the coefficient of $\Theta^+ \in HF^+(S^3, s)$ in the expression $F_{W,s}^{\text{mix}}(U^n \cdot \Theta_- \otimes \zeta)$, where $\Theta_- \in HF^-(S^3, s_0)$ resp. $\Theta_+ \in HF^+(S^3, s_0)$ is a generator whose degree is maximal resp. minimal. Of course, $\Phi_{X,s}$ vanishes on those homogeneous elements whose degree is different from

$$d(s) = \frac{c_1(s)^2 - 2\chi(X) - 3\sigma(X)}{4}.$$

Theorem 9.1. *The map*

$$\Phi_{X,s}: \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{Tors}) \longrightarrow \mathbb{Z}/\pm 1$$

is a smooth, oriented four-manifold invariant. In particular, if $f: X_1 \rightarrow X_2$ is an orientation-preserving diffeomorphism, then

$$\Phi_{X_1, f^*(s)}(U^n \otimes \zeta) = \Phi_{X_2, s}(U^n \otimes f_*(\zeta)).$$

Proof. This is an immediate consequence of Theorem 8.5. \square

10. Properties of the closed four-manifold invariant

We now return to the closed four-manifold invariant introduced in Section 9. We first state a principle which is obvious from the definition of the invariant, and which implies Theorem 1.3. We then turn to the adjunction inequality stated in Theorem 1.5

10.1. Vanishing theorems

Theorem 10.1. *Let Y be a rational homology three-sphere and $t \in \text{Spin}^c(Y)$ with $HF_{\text{red}}(Y, t) = 0$. Suppose that X is a smooth, closed, oriented four-manifold which admits a decomposition $X = X_1 \#_Y X_2$, with $b_2^+(X_1), b_2^+(X_2) > 0$. Then, for each $s \in \text{Spin}^c(X)$ with $s|_Y = t$, we have that $\Phi_{X,s} = 0$.*

Proof. The hypotheses guarantee that the cut along Y is admissible in the sense of Definition 8.3. In particular, the map F^{mix} which is used to define $\Phi_{X,s}$ (i.e. the

mixed invariant associated to the cobordism obtained by puncturing X) factors through $HF_{\text{red}}(Y, \mathfrak{t}) = 0$, so it must vanish. \square

Since $HF_{\text{red}}(S^3) = 0$, the above theorem gives Theorem 1.3.

Observe that the blow-up formula for the cobordism invariants can be rephrased for the absolute invariant, as well. The following is a restatement of Theorem 1.4:

Theorem 10.2. *Let X be a closed, smooth, four-manifold with $b_2^+(X) > 1$, and let $\widehat{X} = X \# \overline{\mathbb{CP}}^2$ be its blowup. Then, for each Spin^c structure $\widehat{\mathfrak{s}} \in \text{Spin}^c(\widehat{X})$, with $d(\widehat{X}, \widehat{\mathfrak{s}}) \geq 0$ we have the relation*

$$\Phi_{\widehat{X}, \widehat{\mathfrak{s}}}(U^{\frac{\ell(\ell+1)}{2}} \cdot \zeta) = \Phi_{X, \mathfrak{s}}(\zeta),$$

where \mathfrak{s} is the Spin^c structure over X which agrees over $X - B^4$ with the restriction of $\widehat{\mathfrak{s}}$, $\zeta \in \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{Tors})$ is any element of degree $d(X, \mathfrak{s})$, and ℓ is calculated by $\langle c_1(\widehat{\mathfrak{s}}), [E] \rangle = \pm(2\ell + 1)$, where $E \subset \widehat{X}$ is the exceptional sphere.

Proof. Suppose that N is an allowable cut for $X = X_1 \#_N X_2$. Then, we can decompose $\widehat{X} = X_1 \#_N X_2 \#_{S^3} \overline{\mathbb{CP}}^2$, and still use N as the cut. The theorem then follows from the composition law for the cobordism $X_2 \# (\overline{\mathbb{CP}}^2 - B^4)$, together with the blowup formula for the maps of cobordisms (Theorem 3.7). \square

10.2. Adjunction inequalities

We turn to a proof of Theorem 1.5.

Proof of Theorem 1.5. As is standard practice, we can reduce to the case where $\Sigma \cdot \Sigma = 0$ with the help of the blowup-formula.

Since we have assumed that $b_2^+(X) > 1$, we can find an admissible cut N which is disjoint from Σ . Specifically, we can find another smoothly embedded surface $T \subset X$ with positive self-intersection number, which is disjoint from Σ , and use its tubular neighborhood as the admissible cut. Thus, letting W be complement in X of a pair of balls, we $W = W_1 \cup W_2$, where W_2 contains the embedded surface Σ . Now, the map

$$F_{W_2, \mathfrak{s}|_{W_2}}^+ : HF^+(N, \mathfrak{s}|_N) \longrightarrow HF^+(S^3)$$

factors through $\underline{HF}^+(S^1 \times \Sigma, \mathfrak{s}|_{S^1 \times \Sigma}, M)$, where we take $S^1 \times \Sigma \subset X_2$ to be the tubular neighborhood of Σ , and M to be some suitable twisted coefficient system (as in Theorem 3.9). The adjunction inequality for the three-manifold $S^1 \times \Sigma$ shows that this group is zero, unless

$$|\langle c_1(\mathfrak{s}), [\Sigma] \rangle| < 2g - 2. \quad \square$$

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